Renormalization and Improvement

Lecture 3

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EuroPLEx Summer School 2021 on
Lattice Field Theory and Applications

University of Edinburgh

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EUROPLEX
Summer School 2021 on lattice field theory and applications

Renormalization in lattice QCD

K. Wilson contributed significantly in the renormalization since the 1960s

"*The origins of Lattice Gauge Theories*",
Lattice 2004, arXiv:hep-lat/0412043

Renormalization removed divergence as well as dependence on lattice regulator, so that the continuum limit ($a \to 0$) can be recovered

Perturbative Renormalization

RI-MOM scheme

★ Regularization Independent momentum subtraction (RI-MOM) schemes naturally defined in perturbation theory

★ Calculation of Green functions of operators at given off-shell external states with momentum p

★ A condition is applied on the Green functions to match them with their tree-level value

★ Examples of RI-type conditions:

For fermion field

$$Z_q^{\rm RI} = \frac{1}{12} {\rm Tr}\left[(S^L)^{-1}(p)\, S^{\rm tree}(p)\right]\Big|_{p^2=\mu^2}$$

For operator $\bar\psi\Gamma\psi$

$$(Z_q^{\rm RI})^{-1}\, Z_\Gamma^{\rm RI}\, {\rm Tr}\left[G_\Gamma^L(p)\, G_\Gamma^{\rm tree}(p)\right]\Big|_{p^2=\mu^2} = {\rm Tr}\left[G_\Gamma^{\rm tree}(p)\, G_\Gamma^{\rm tree}(p)\right]\Big|_{p^2=\mu^2}$$

Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

★ Starting point: fermion propagators in momentum space

$$S^u(p) = \frac{a^8}{V} \sum_{x,y} e^{-ip(x-y)} \langle u(x)\overline{u}(y)\rangle \ , \qquad S^d(p) = \frac{a^8}{V} \sum_{x,y} e^{-ip(x-y)} \langle d(x)\overline{d}(y)\rangle$$

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```
martha-imac:prop martha_laptop$ more p_propagator_p2_2_2_2.0500.dat
d 0 0 0 0 2.601862e-02 -5.076038e-01
u 0 0 0 0 2.601862e-02 5.076038e-01
d 0 0 0 1 1.233905e-03 -1.207621e-03
u 0 0 0 1 -1.496412e-04 3.169783e-03
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Conf

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Conf

momentum indices

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```

$$ap = 2\pi\left(\frac{n_t+\frac{1}{2}}{T}, \frac{n_x}{L}, \frac{n_y}{L}, \frac{n_z}{L}\right)$$

Here:

$n_t = 2, n_x = 2, n_y = 2, n_z = 2$

$$L^3 \times T = 24^3 \times 48$$

$$N_f = 4$$

$\beta = 1.726,\quad a = 0.093$ fm		
$a\mu$	am_{PS}	lattice size
0.0060	0.1680	$24^3 \times 48$

Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

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Conf

d-quark

u-quark

momentum indices

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Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

p_propagator_p2_2_2_2_0500.dat

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d 0 1 1 0 -1.063519e-03 1.658460e-03
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Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

s_1, s_2, c_1, c_2

p_propagator_p2_2_2_2_0500.dat

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u 0 0 0 0 2.601862e-02 5.076038e-01
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```

Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

p_propagator_p2_2_2_2_0500.dat

	s_1,	s_2,	c_1,	c_2	Real	Imaginary
d	0	0	0	0	2.601862e-02	-5.076038e-01
u	0	0	0	0	2.601862e-02	5.076038e-01
d	0	0	0	1	1.233905e-03	-1.207621e-03
u	0	0	0	1	-1.496412e-04	3.169783e-03
d	0	0	0	2	-1.310907e-03	-2.694817e-03
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Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

★ Each value of momentum is analyzed independently, e.g., p“=”(2,2,2,2) for all the configurations (10)

★ Use the data for up-quark or down-quark separately.
Z_q is extracted from either of these

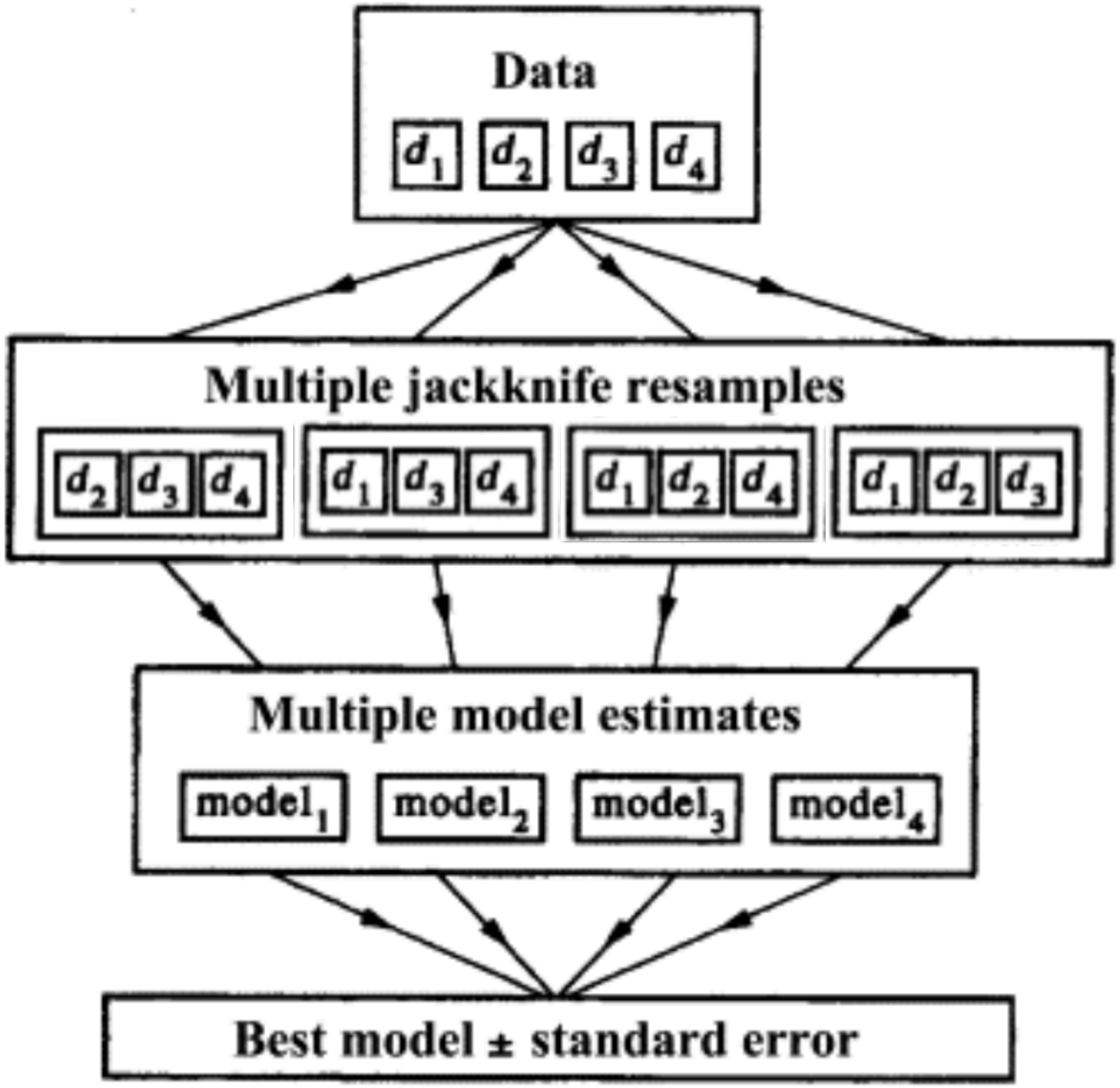
★ Propagator is a 12 x 12 matrix (spin-color space). You can use two unique indices instead of 4: 3*s1+c1,3*s2+c2

★ A jackknife analysis is applied on the propagators to find the “bins”

Error estimation

Results MUST be accompanied by uncertainties

Jackknife resampling for variance and bias estimation



★ Choose the number of omitted data in each bin (defines # bins)

★ Calculate the average over remaining data in each bin

★ Calculate the average of the bins

★ Calculate the statistical error of the above average

Error estimation

Results MUST be accompanied by uncertainties

Jackknife resampling
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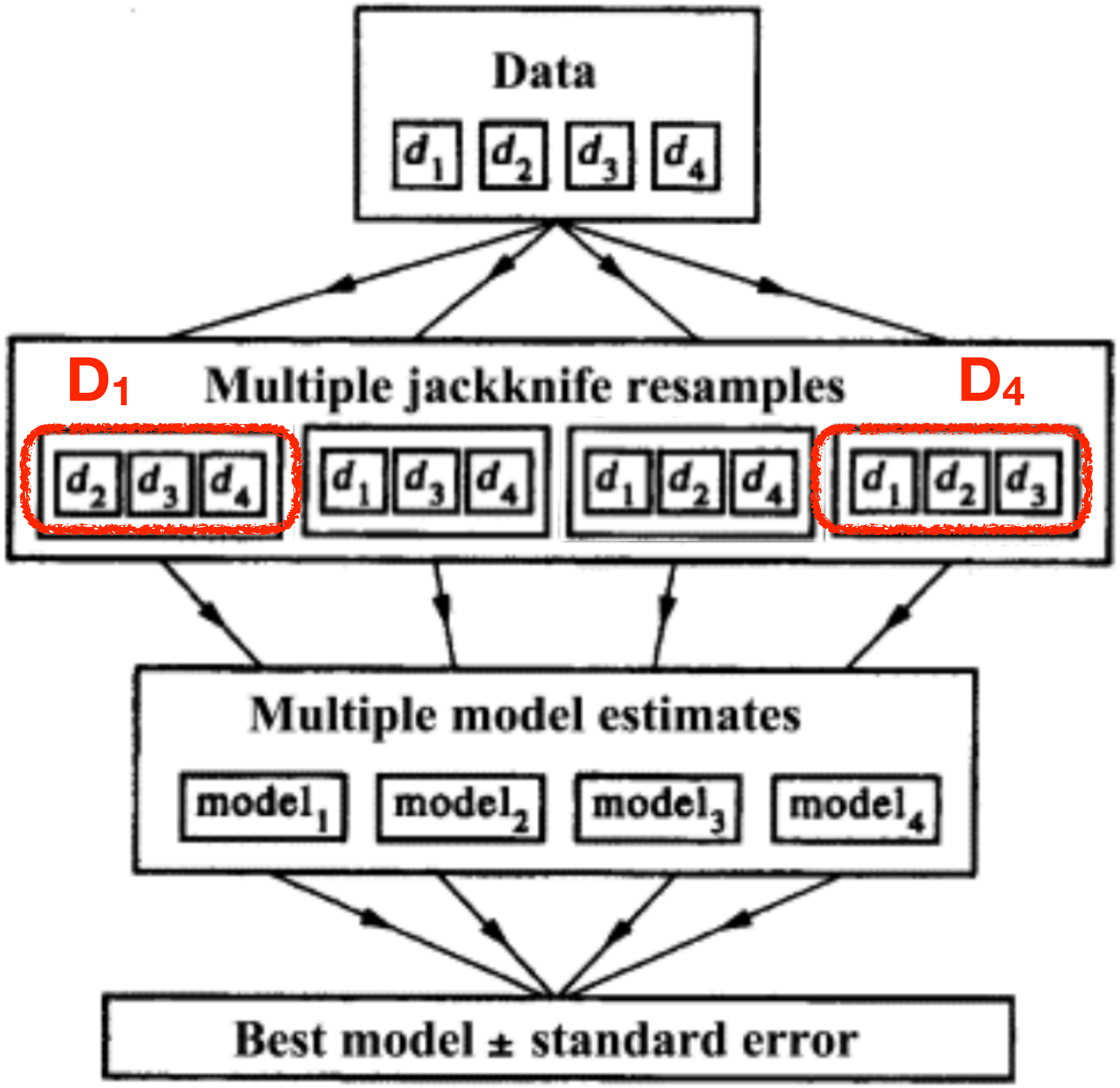

- ★ Choose the number of omitted data in each bin (defines # bins)
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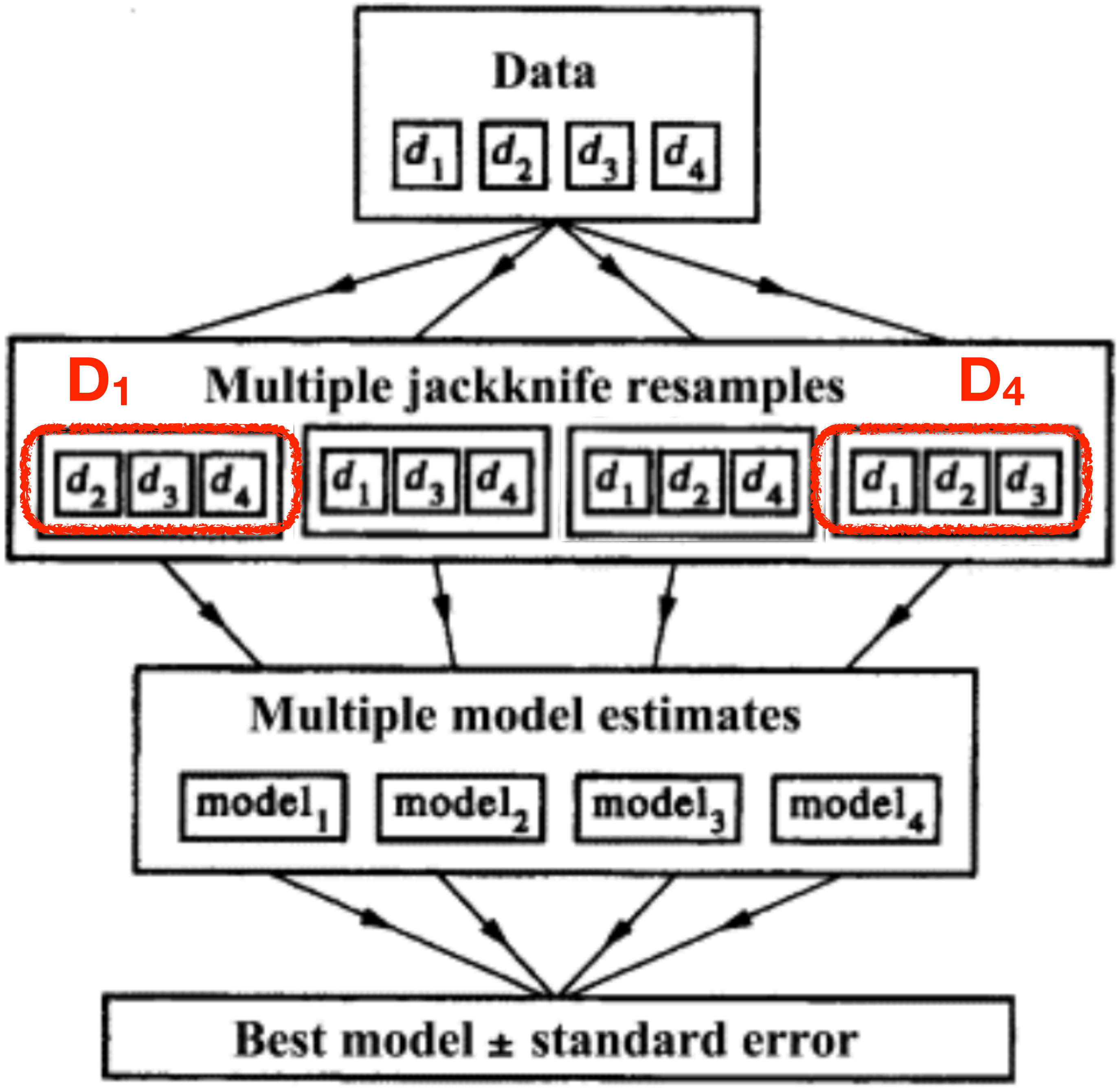

- ★ Choose the number of omitted data in each bin (defines # bins)

$$N_{\text{data}} = 4, \;\; N_{\text{omit}} = 1\,, \;\; N_{\text{bin}} = 4$$

- ★ Calculate the average over remaining data in each bin
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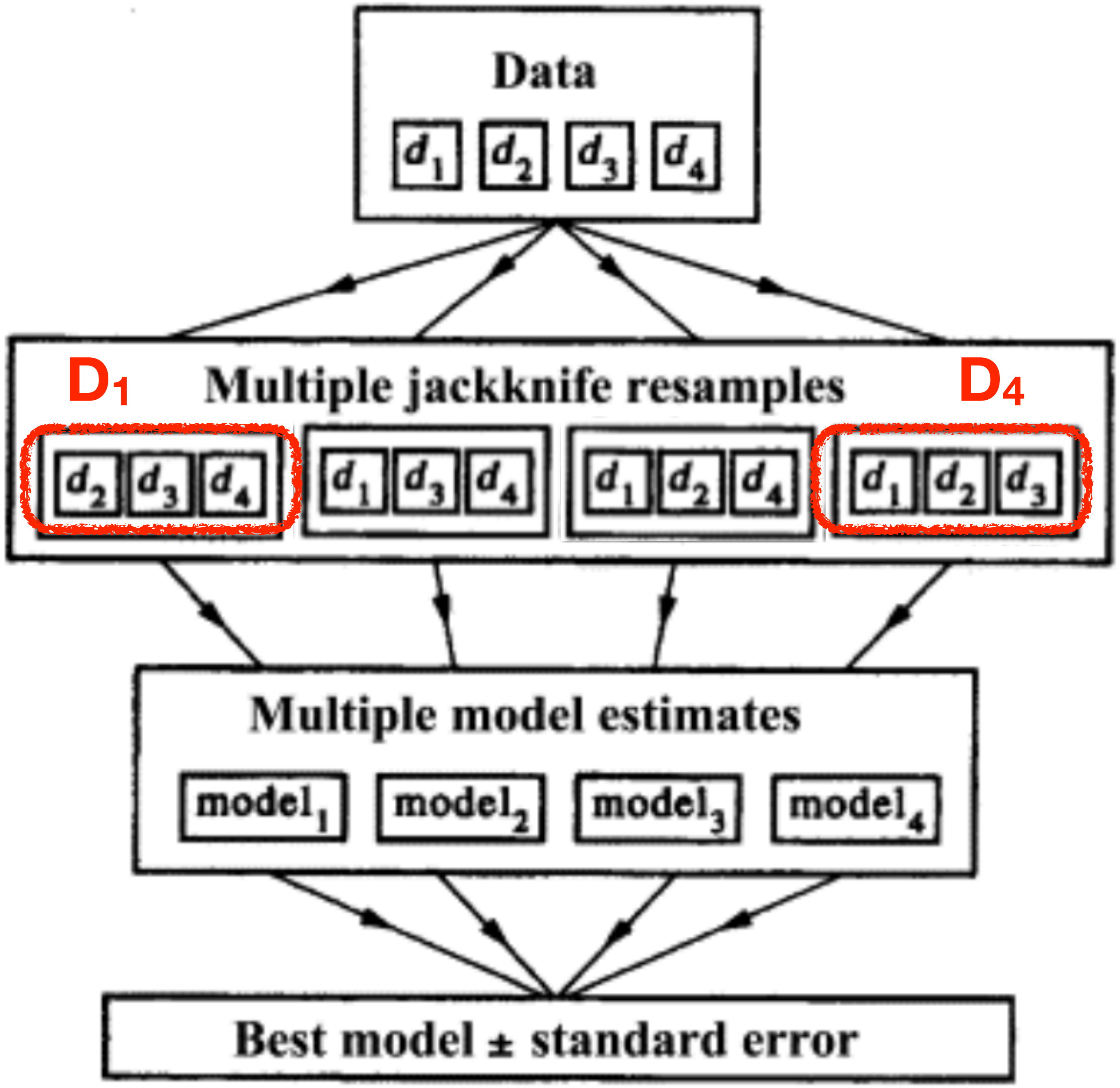

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$$N_{\text{data}} = 4, \quad N_{\text{omit}} = 1\,, \quad N_{\text{bin}} = 4$$

- ★ Calculate the average over remaining data in each bin

$$D_{\text{i}} = \sum_{j \neq i} \frac{d_j}{N_{\text{data}} - N_{\text{omit}}}$$

- ★ Calculate the average of the bins

- ★ Calculate the statistical error of the above average

Error estimation

Results MUST be accompanied by uncertainties

Jackknife resampling
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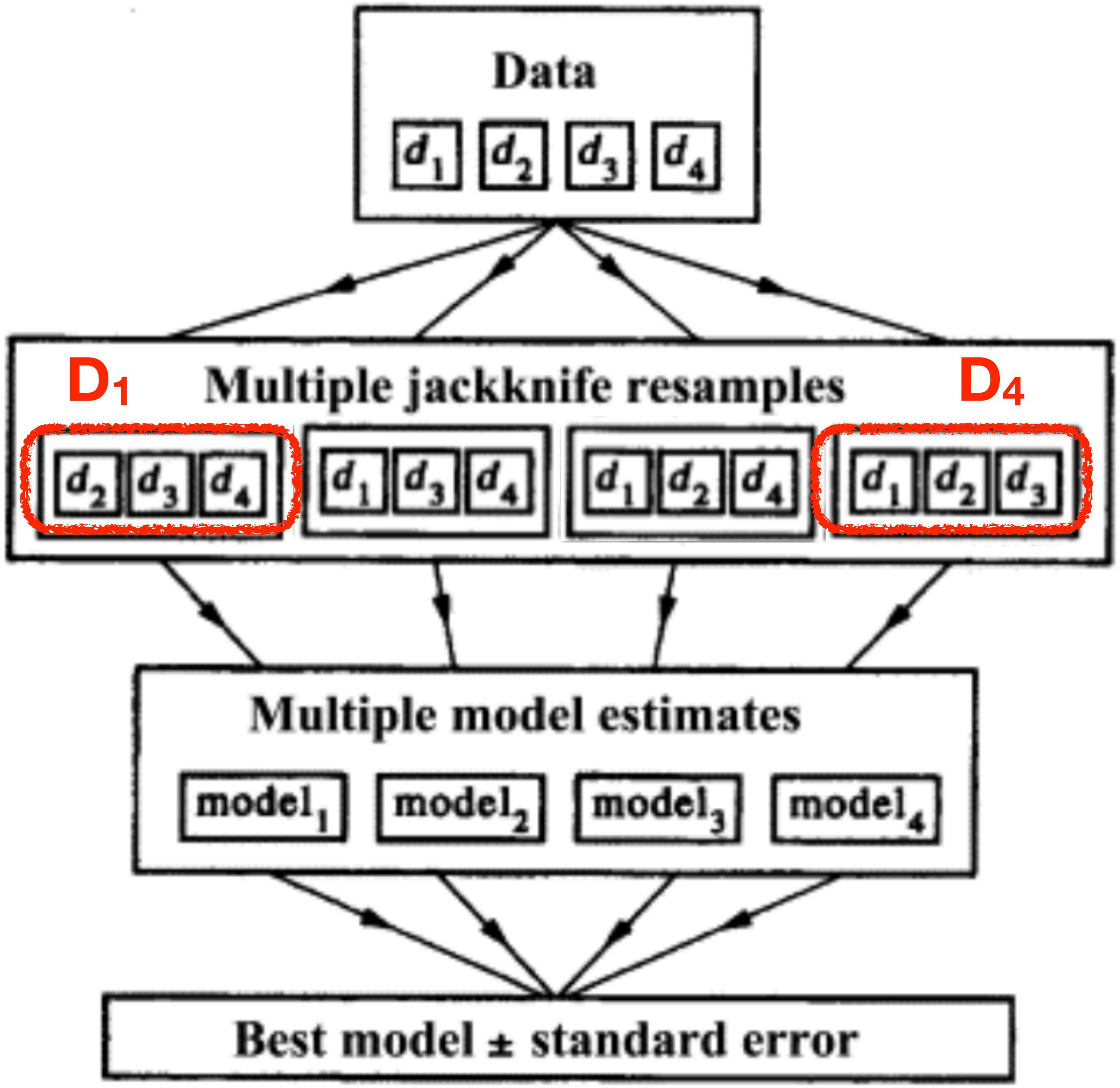

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$$\bar{D} = \sum_{i} \frac{D_{\text{i}}}{N_{\text{bin}}}$$

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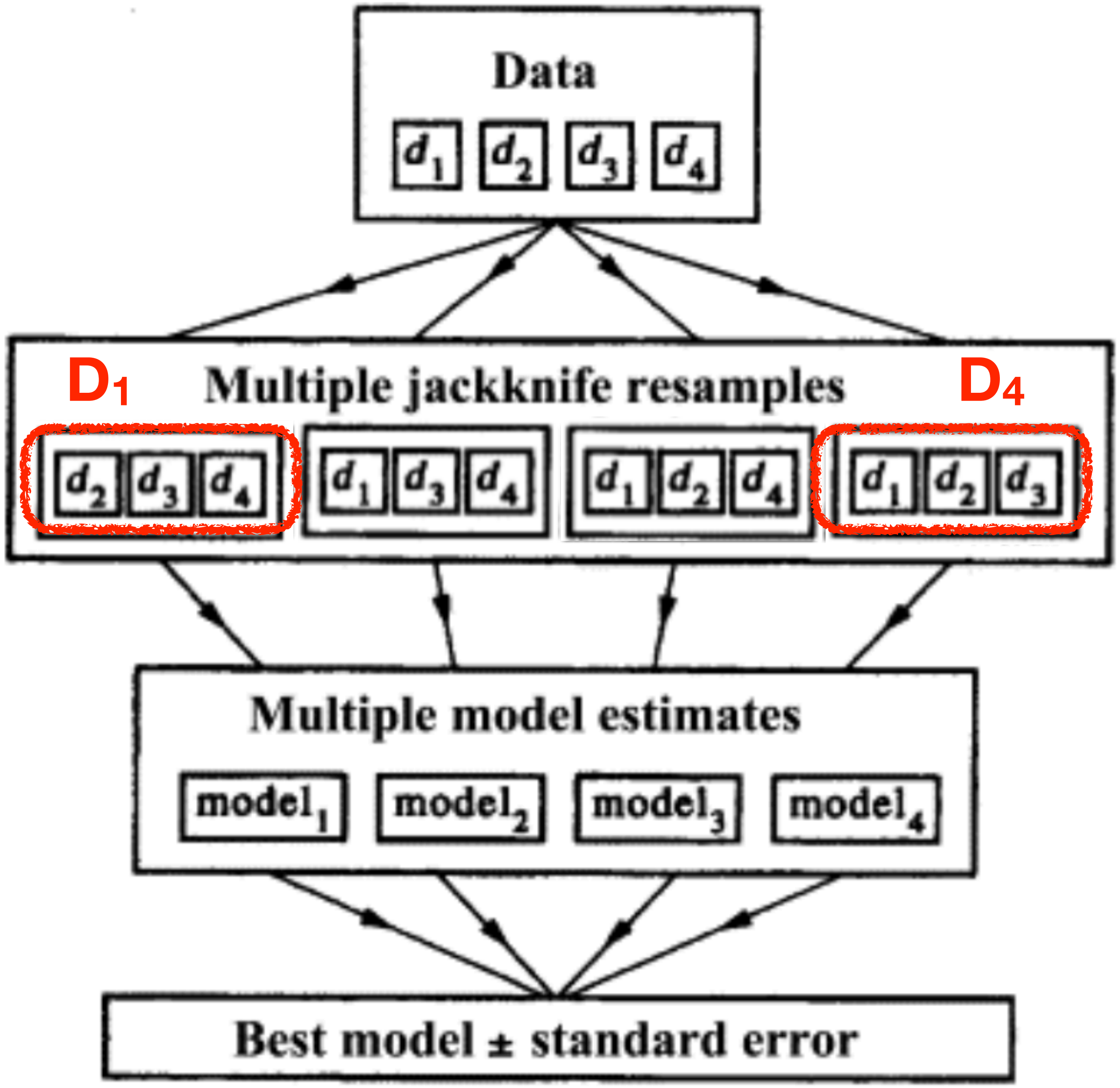
Jackknife resampling for variance and bias estimation



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$$N_{\text{data}} = 4, \quad N_{\text{omit}} = 1\,, \quad N_{\text{bin}} = 4$$

- ★ Calculate the average over remaining data in each bin

$$D_{\text{i}} = \sum_{j \neq i} \frac{d_j}{N_{\text{data}} - N_{\text{omit}}}$$

- ★ Calculate the average of the bins

$$\bar{D} = \sum_i \frac{D_{\text{i}}}{N_{\text{bin}}}$$

- ★ Calculate the statistical error of the above average

$$d\bar{D} = \sqrt{\sum_i (D_i - \bar{D})^2} \sqrt{\frac{N_{\text{bin}} - 1}{N_{\text{bin}}}}$$

Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

★ On each bin, take the inverse of the propagator

★ On each bin, apply the renormalization prescription

$$Z_q = \frac{1}{12}\mathrm{Tr}\left[\boxed{\frac{-i\sum_\rho \gamma_\rho \sin(p_\rho)}{\sum_\rho \sin(p_\rho)^2}}\, S_q^{-1}(p)(p)\right]\Bigg|_{p^2=\mu^2}$$

Tree-level (diagonal in color space)

where $ap = 2\pi\left(\frac{n_t+\frac{1}{2}}{T}, \frac{n_x}{L}, \frac{n_y}{L}, \frac{n_z}{L}\right)$

Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

★ Matrices basis

Row 1

```
[marthac@ela3 beta_1.726]$ more matrices.txt
1  0     0  0     0  0     0  0     0  0      1  0
0  0     0  0     0  0     0  0     0  0      0  0
0  0    -1  0     0  0     0  0     0 -1      0  0
0  0     0  0     0 -1    -1  0     0  0      0  0
0  0     0  0     0  0     0  0     0  0      0  0
1  0     0  0     0  0     0  0     0  0      1  0
0  0     0  0     0 -1     1  0     0  0      0  0
0  0    -1  0     0  0     0  0     0  1      0  0
0  0    -1  0     0  0     0  0     0  1      0  0
0  0     0  0     0  1     1  0     0  0      0  0
1  0     0  0     0  0     0  0     0  0     -1  0
0  0     0  0     0  0     0  0     0  0      0  0
0  0     0  0     0  1    -1  0     0  0      0  0
0  0    -1  0     0  0     0  0     0 -1      0  0
0  0     0  0     0  0     0  0     0  0      0  0
1  0     0  0     0  0     0  0     0  0     -1  0
```

1, γ_0, γ_1, γ_2, γ_3, γ_5

Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

★ Once you have the bin values of Z_q calculate the average and the jackknife error.

★ The estimate for Z_q corresponds to a renormalization scale $(ap)^2$. Therefore, different estimate Z_q is expected for each momentum

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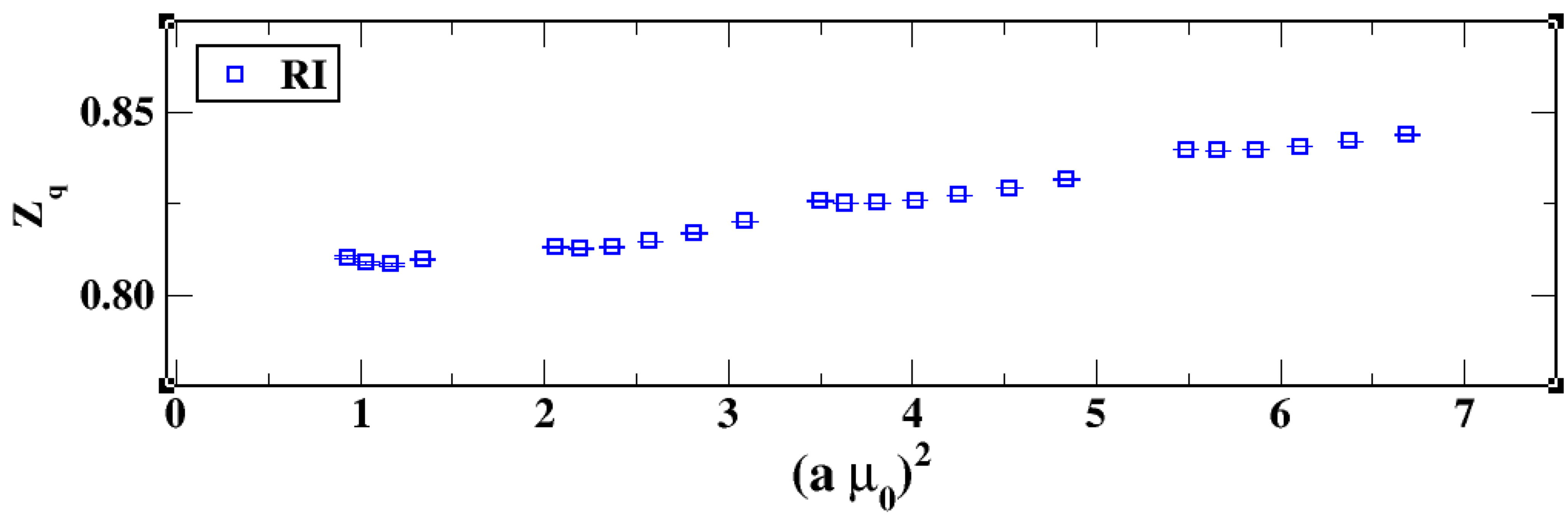

Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

★ The scheme and scale dependence in the physical matrix elements,

e.g. $\langle N | \bar{\Psi}\, \Gamma\, \Psi | N \rangle$

is introduced through the renormalization $Z_\Gamma^S(\mu)\, \langle N | \bar{\Psi}\, \Gamma\, \Psi | N \rangle$

★ To compare with experimental data, one needs to convert to a convenient scheme and renormalization scale.

The necessary conversion and evolution is applied on $Z_\Gamma^S(\mu)$

(by definition, the scheme & scale are introduced during the renormalization procedure)

★ Typically one chooses the $\overline{\mathrm{MS}}$ scheme at a scale μ of 2 GeV

Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

★ For the scheme conversion it is convenient to use the intermediate Renormalization Group Invariant (RGI) scheme, which is scale independent and relates the RI′ and $\overline{\mathrm{MS}}$ results

$$Z_{\mathcal{O}}^{\mathrm{RGI}} = Z_{\mathcal{O}}^{\mathrm{RI}'}(\mu)\,\Delta Z_{\mathcal{O}}^{\mathrm{RI}'}(\mu) = Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV})\,\Delta Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV})$$

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★ The conversion factor can be read from the above relation

$$Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) \equiv C_{\mathcal{O}}^{\mathrm{RI}',\overline{\mathrm{MS}}}(\mu, 2\,\mathrm{GeV})\, Z_{\mathcal{O}}^{\mathrm{RI}'}(\mu)\,, \qquad C_{\mathcal{O}}^{\mathrm{RI}',\overline{\mathrm{MS}}}(\mu, 2\,\mathrm{GeV}) = \frac{\Delta Z_{\mathcal{O}}^{\mathrm{RI}'}(\mu)}{\Delta Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV})}$$

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$$\Delta Z_{\mathcal{O}}^{\mathcal{S}}(\mu) = \left(2\beta_0 \frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2}\right)^{-\frac{\gamma_0}{2\beta_0}} \exp\left\{\int_0^{g^{\mathcal{S}}(\mu)} \mathrm{d}g' \left(\frac{\gamma^{\mathcal{S}}(g')}{\beta^{\mathcal{S}}(g')} + \frac{\gamma_0}{\beta_0\, g'}\right)\right\}$$

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$\Lambda_{\overline{\mathrm{MS}}} = 294$ (Nf=4)

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$$\begin{aligned} \beta_0 &= 11 - \frac{2}{3} N_f\,, \\ \beta_1 &= 102 - \frac{38}{3} N_f\,, \\ \beta_2 &= \frac{2857}{2} - \frac{5033}{18} N_f + \frac{325}{54} N_f^2 \end{aligned}$$

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★ To 3-loop approximation:

$$\Delta Z^{\mathcal{S}}_{\mathcal{O}}(\mu) = \left(2\beta_0 \frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2}\right)^{-\frac{\gamma_0}{2\beta_0}} \left(1 + \frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} \frac{\beta_1\gamma_0 - \beta_0\gamma_1^{\mathcal{S}}}{2\beta_0^2} + \frac{g^{\mathcal{S}}(\mu)^4}{(16\pi^2)^2} \frac{-2\beta_0^3\gamma_2^{\mathcal{S}} + \beta_0^2(\gamma_1^{\mathcal{S}}(2\beta_1 + \gamma_1^{\mathcal{S}}) + 2\beta_2\gamma_0) - 2\beta_0\beta_1\gamma_0(\beta_1 + \gamma_1^{\mathcal{S}}) + \beta_1^2\gamma_0^2}{8\beta_0^4}\right)$$

$$\frac{g^{\overline{\mathrm{MS}},\mathrm{RI}'}(\mu)^2}{16\pi^2}\bigg|_{3-\mathrm{loop}} = \frac{1}{\beta_0\, L} - \frac{\beta_1}{\beta_0^3} \frac{\log L}{L^2} + \frac{1}{\beta_0^5} \frac{\beta_1^2 \log^2 L - \beta_1^2 \log L + \beta_2\beta_0 - \beta_1^2}{L^3}), \quad L = \log \frac{\mu^2}{\Lambda^2_{\overline{\mathrm{MS}}}}$$

Here

$\Lambda_{\overline{\mathrm{MS}}} = 294$ (Nf=4)

Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

- Each value is converted to the $\overline{\text{MS}}$ scheme and evolved to a common scale (2 GeV)

$$Z_q^{\overline{\text{MS}}}(2\,\text{GeV}) = C^{\overline{\text{MS}},\,\text{RI}}(2\,\text{GeV}, \mu_0)\, \mathcal{Z}_q^{\text{RI}}(\mu_0)$$

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RI-scale				$C^{\overline{\text{MS}},\text{RI}}(2\,\text{GeV}, \mu_0)$
2	2	2	2	0.99100206897122278
3	2	2	2	0.99294282316067028
3	3	3	3	1.0032896646629899
3	4	4	4	1.0091758085180254
4	2	2	2	0.99510205432872045
4	3	3	3	1.0040829703706433
4	4	4	4	1.0095521548715289
4	5	5	5	1.0132348431953588
5	2	2	2	0.99730459096929835
5	3	3	3	1.0049781204593131
5	4	4	4	1.0099952827090808
5	5	5	5	1.0134870227807959
6	3	3	3	1.0059351309588331
6	4	4	4	1.0104912752396640
6	5	5	5	1.0137760840862191
7	3	3	3	1.0069205185317565
7	4	4	4	1.0110265994346921
7	5	5	5	1.0140961806241273
8	3	3	3	1.0079082825459709
8	4	4	4	1.0115888346131423
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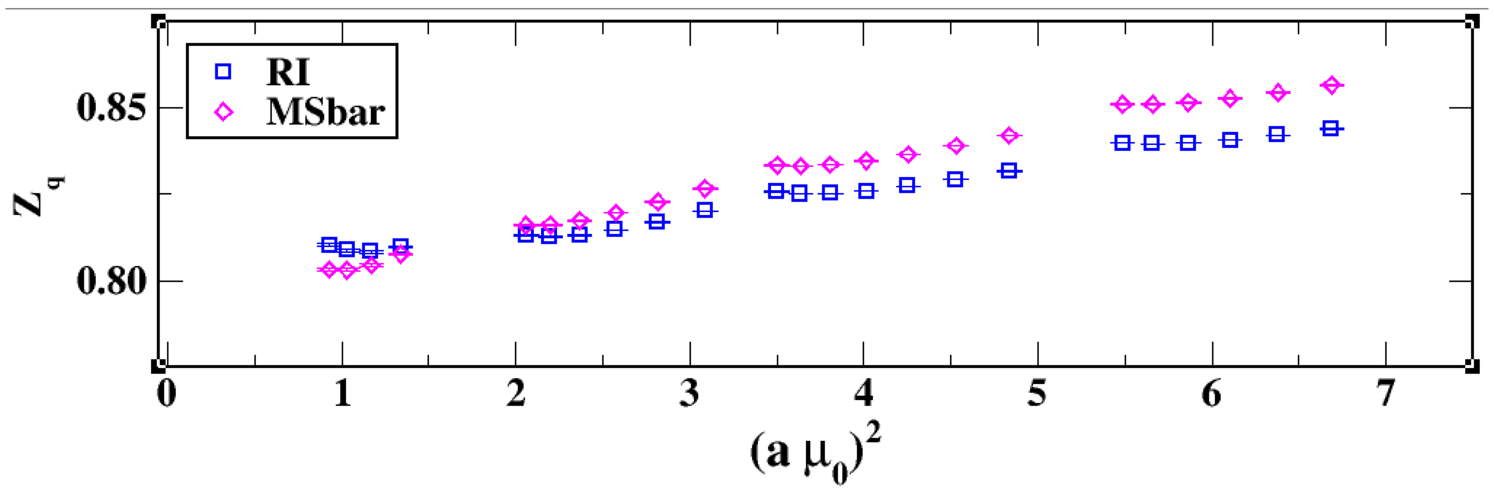

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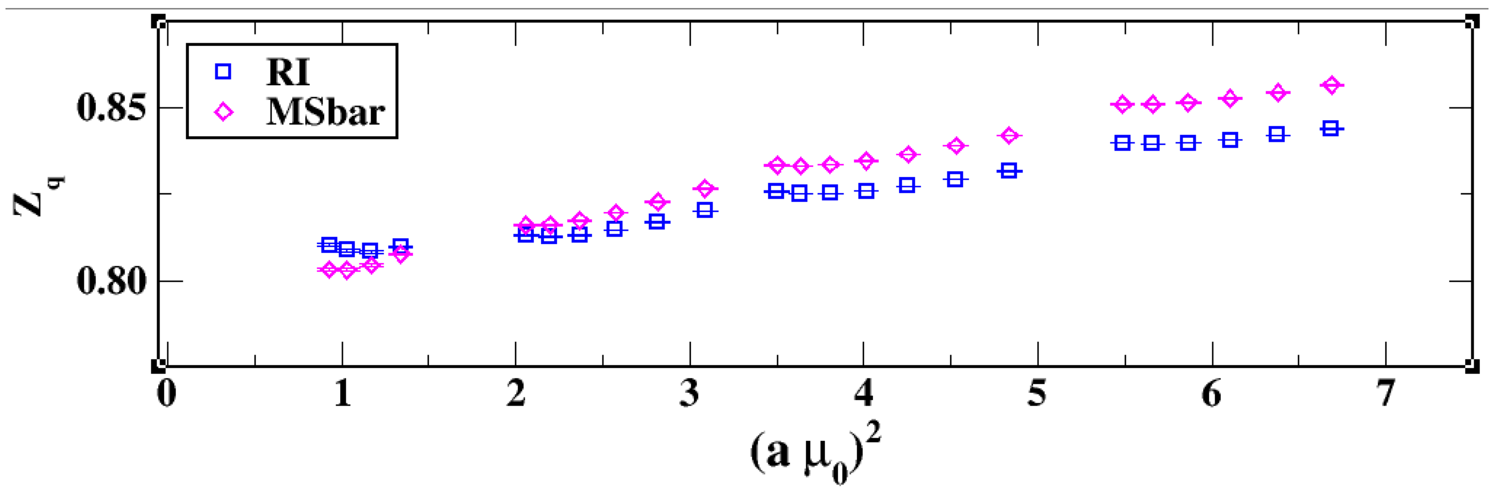

★ Lattice artifacts lead to large slopes

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Contributions of Lattice Pert. Theory

1. Mixing coefficients can be difficult to extract non-perturbatively due to signal-noise problem

2. Renormalization of new operators for which non-perturbative prescriptions are not available

3. Qualitative understanding of the renormalization pattern of operators

4. Operator improvement

5. Synergy between perturbative and non-perturbative results helps improve final estimates for renormalization functions

Contributions of Lattice Pert. Theory (1)

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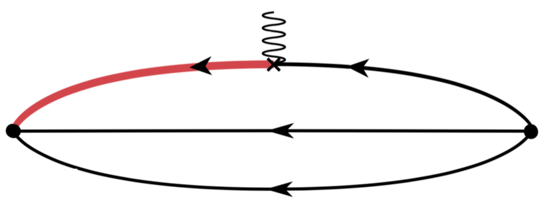
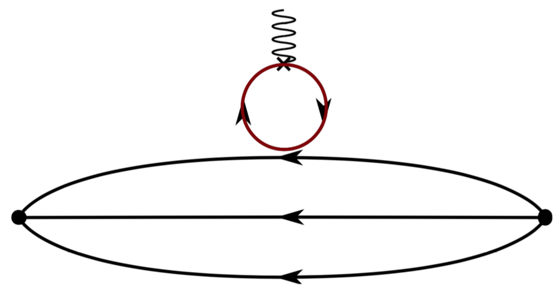
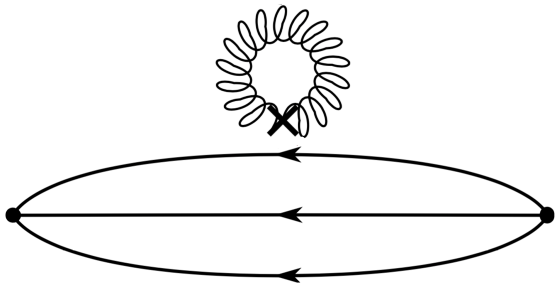


connected



quark disconnected



gluon disconnected

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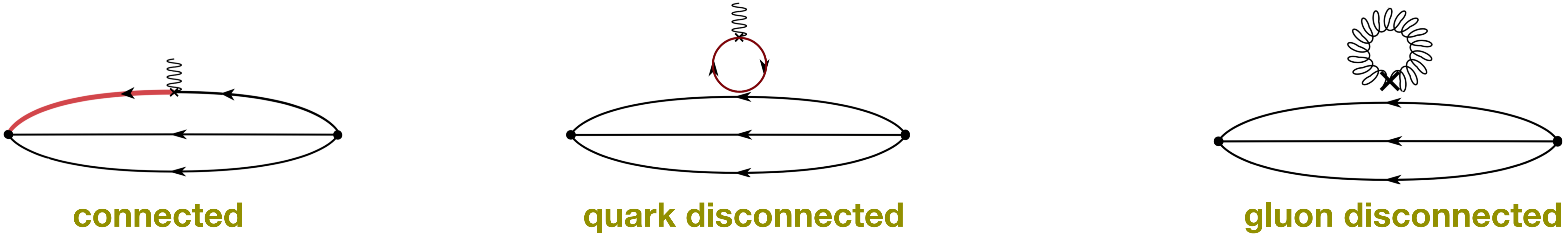

$$\sum_q \langle x \rangle_q^R = Z_{qq} \sum_q \langle x \rangle_q^B$$

$$\langle x \rangle_g^R = Z_{gg} \langle x \rangle_g^B$$

Absence of mixing: multiplicative renormalization

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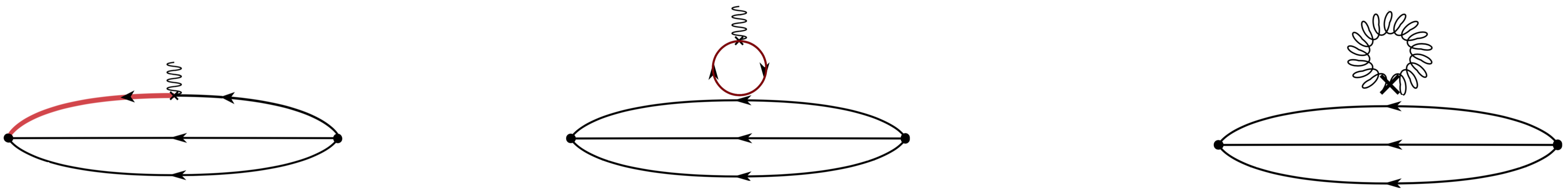

connected　　quark disconnected　　gluon disconnected

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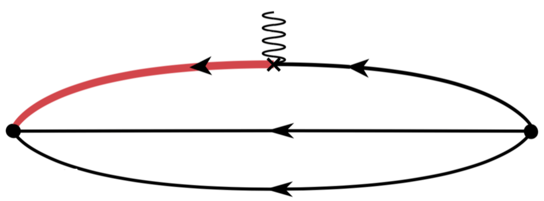
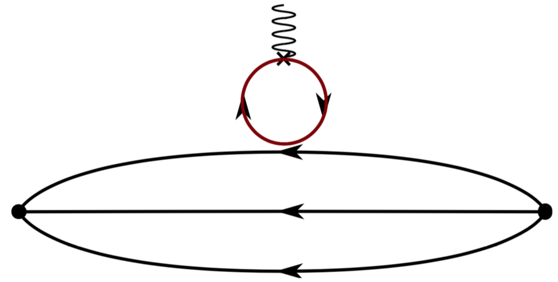
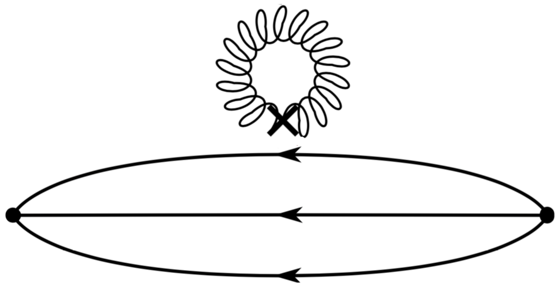


connected



quark disconnected



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Presence of mixing: contributions from multiple matrix elements

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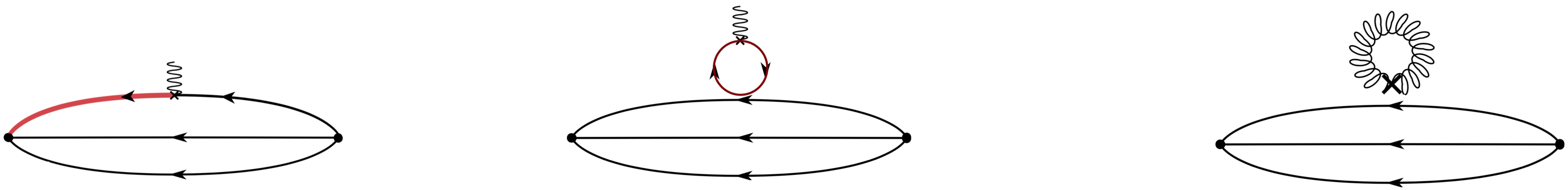

connected   quark disconnected   gluon disconnected

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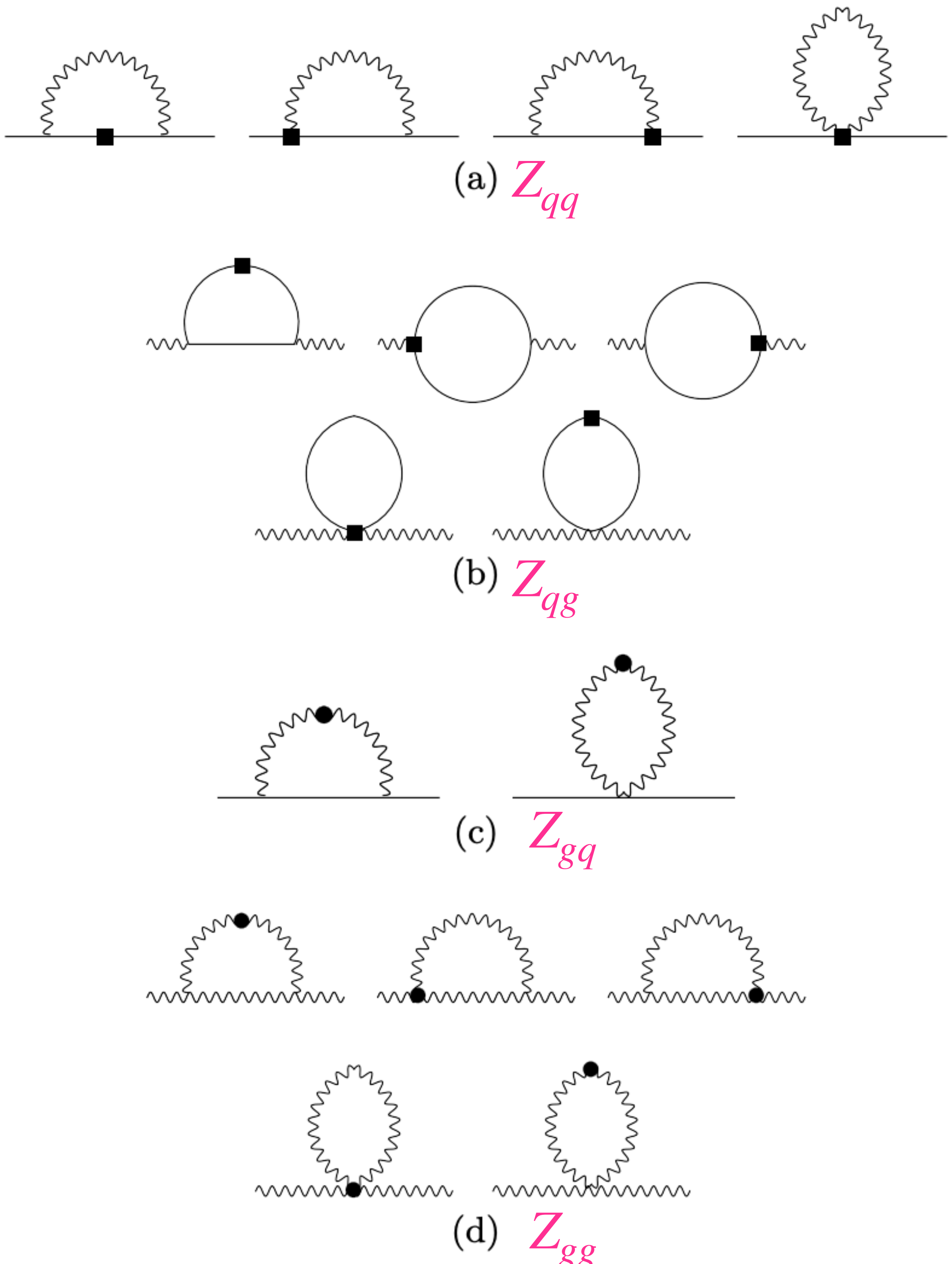
Absence of mixing: multiplicative renormalization

Presence of mixing: contributions from multiple matrix elements

Mixing coefficients Z_{gq}, Z_{qg} :

- subleading compared to Z_{qq}, Z_{gg}
- difficult to extract non-perturbatively due to gauge noise

Contributions of Lattice Pert. Theory (1)



★ Analytic expressions

$$Z_{gg} = 1 + \frac{g^2}{16\pi^2}\left(\frac{e_{11}^{(1)}}{N_c} + e_{11}^{(2)}\, N_f - \frac{2\,N_f}{3}\log(a^2\bar{\mu}^2)\right)$$

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★ Non-perturbative calculations unsuccessful for Z_{gq} and Z_{qg}

[Y.B Yang et al., PRD 98, 074506 (2018), arXiv:1805.00531]

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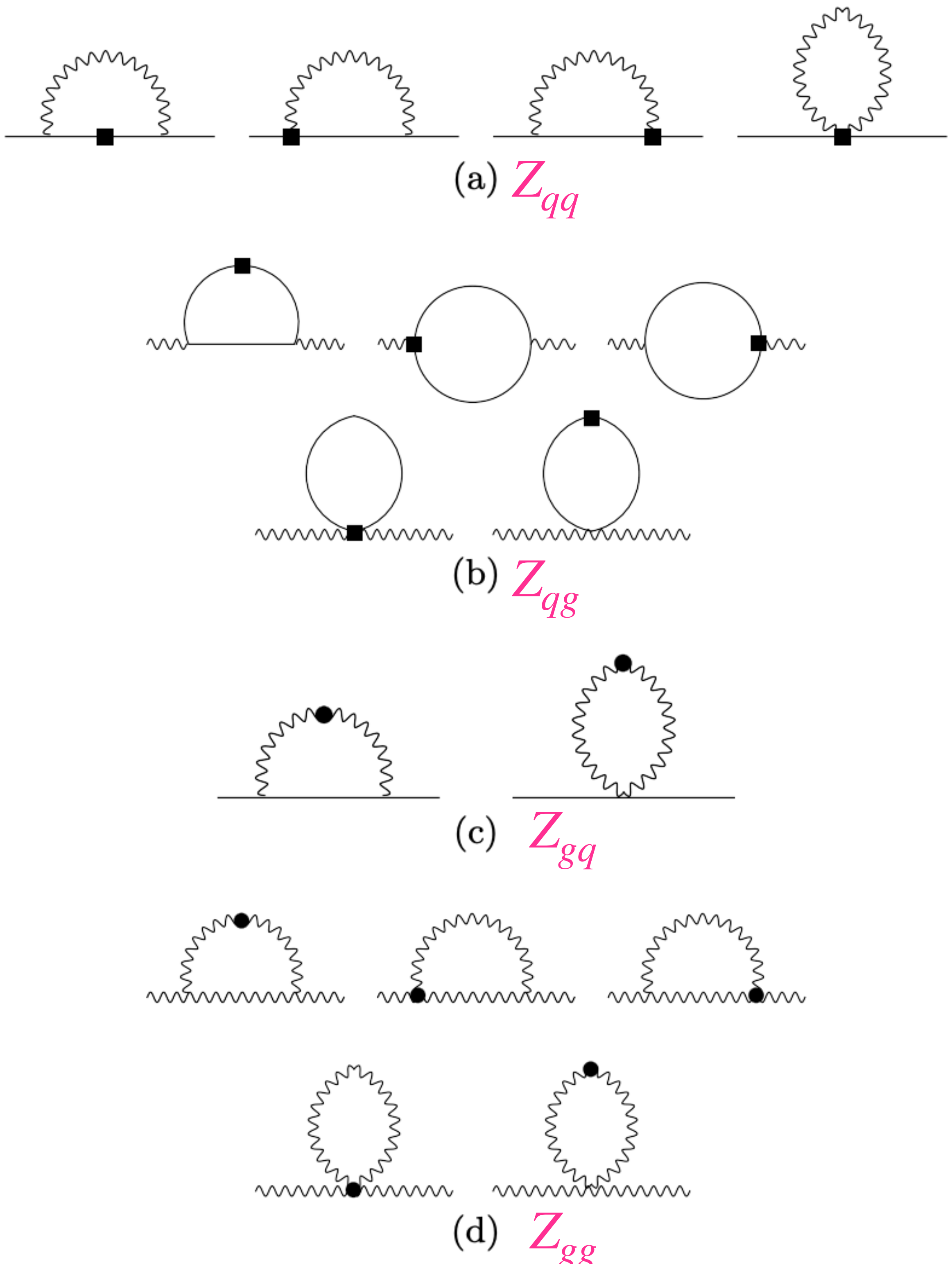
Contributions of Lattice Pert. Theory (1)

Quark and gluon momentum fraction mixing in LPT



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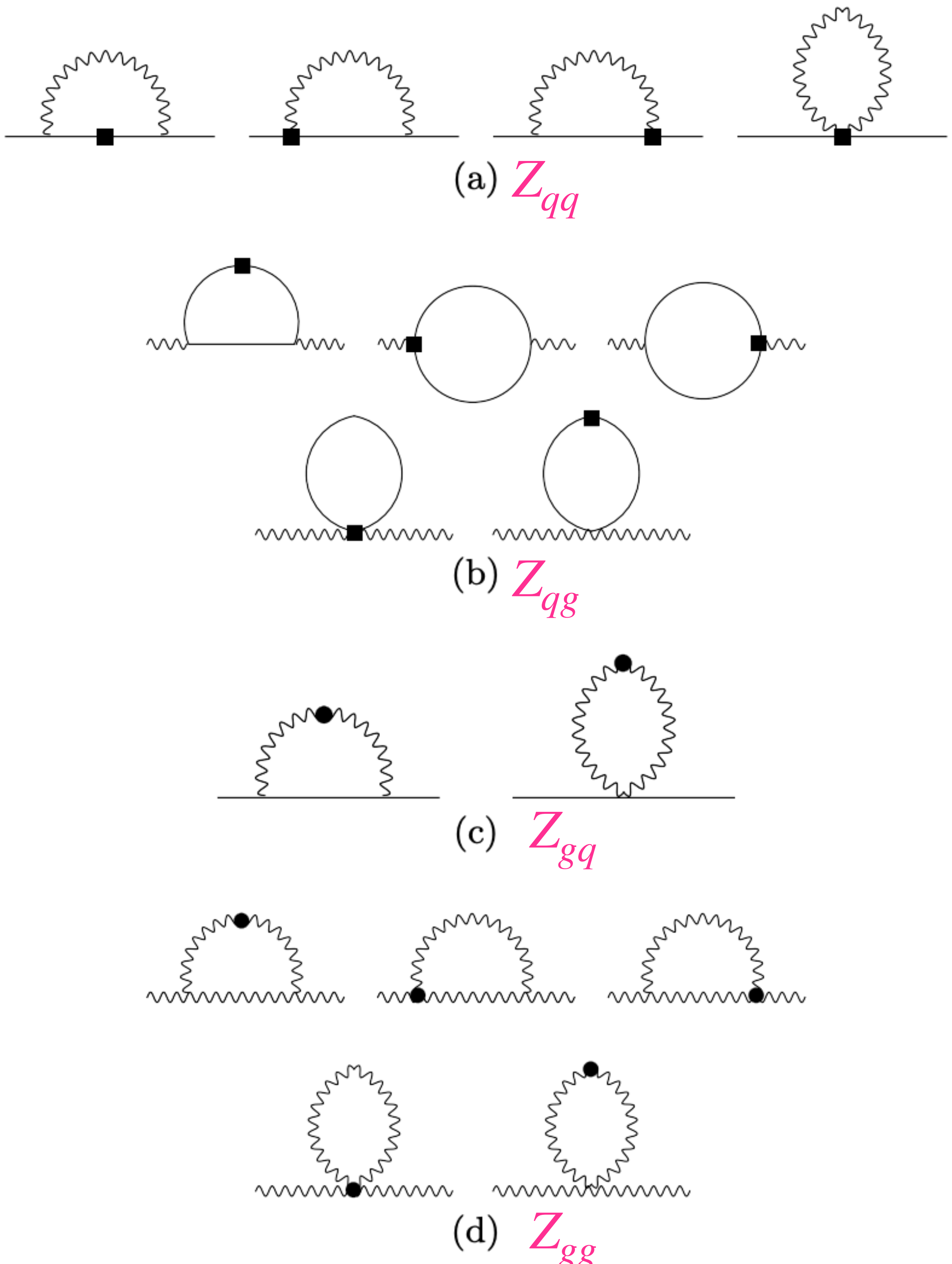


★ Mixing coefficients can be isolated

★ Analytic expressions

$$Z_{gg} = 1 + \frac{g^2}{16\pi^2}\left(\frac{e_{11}^{(1)}}{N_c} + e_{11}^{(2)}\,N_f - \frac{2\,N_f}{3}\log(a^2\bar{\mu}^2)\right)$$
$$Z_{gq} = 0 + \frac{g^2\,C_f}{16\pi^2}\left(e_{12}^{(1)} + e_{12}^{(2)}\,c_{\rm SW} + \frac{8}{3}\log(a^2\bar{\mu}^2)\right).$$

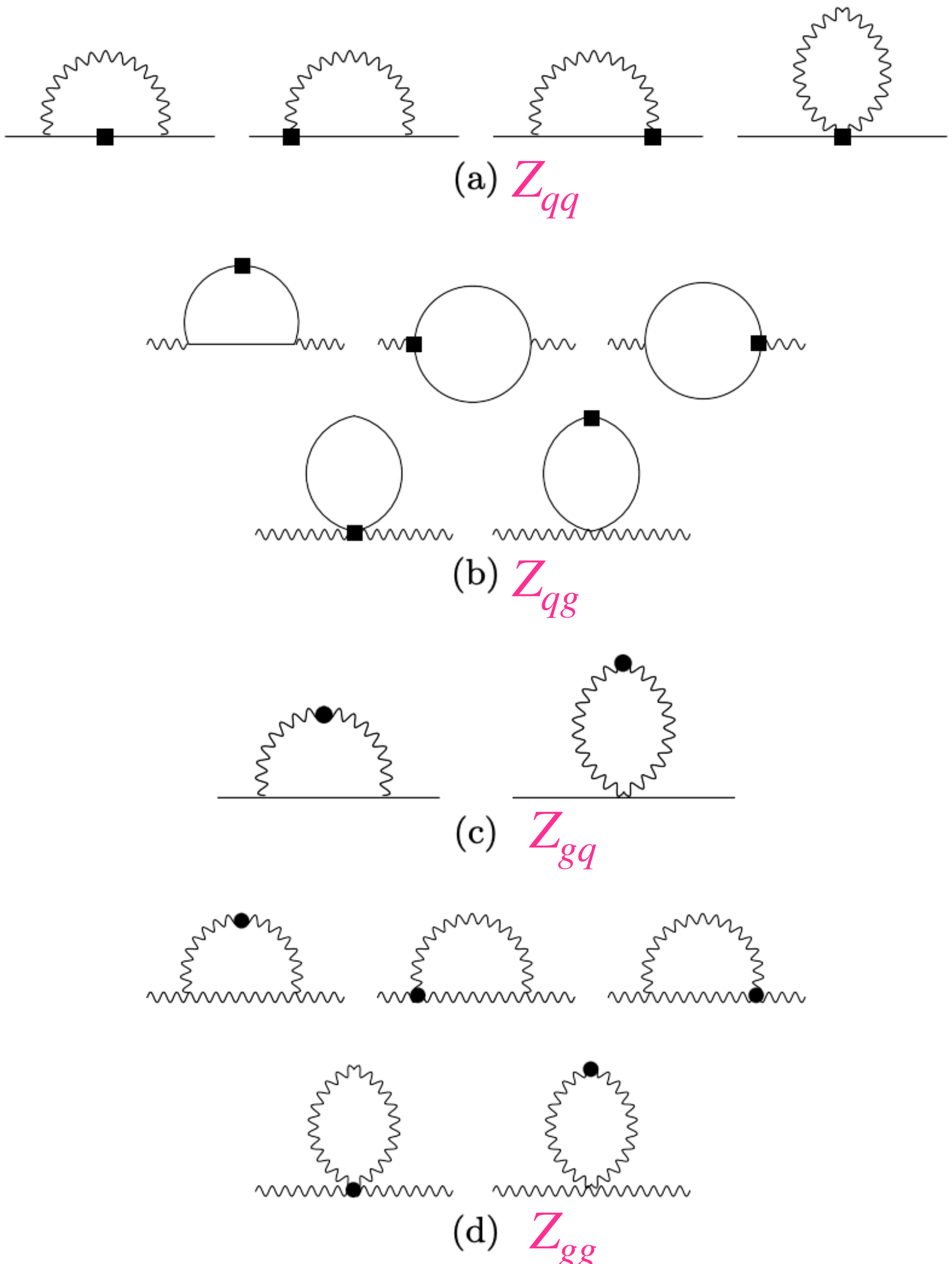
★ Non-perturbative calculations unsuccessful for Z_{gq} and Z_{qg}

[Y.B Yang et al., PRD 98, 074506 (2018), arXiv:1805.00531]

[C. Alexandrou et al., PRD 101, 094513 (2020), arXiv:2003.08486]

Contributions of Lattice Pert. Theory (1)

Quark and gluon momentum fraction mixing in LPT



- ★ Mixing coefficients can be isolated

- ★ Calculation is very taxing! Many millions of terms…

- ★ Analytic expressions

$$Z_{gg} = 1 + \frac{g^2}{16\pi^2}\left(\frac{e_{11}^{(1)}}{N_c} + e_{11}^{(2)}\,N_f - \frac{2\,N_f}{3}\log(a^2\bar{\mu}^2)\right)$$

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