Renormalization and Improvement

Lecture 4

Martha Constantinou

Temple University

EuroPLEx Summer School 2021 on
Lattice Field Theory and Applications

University of Edinburgh

August 26 - September 3, 2021

EUROPLEX
Summer School 2021 on lattice field theory and applications

Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

★ The scale dependence of Z-factors of operators is determined by its anomalous dimension

$$\gamma^{\mathcal{S}} = -\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \log Z_{\mathcal{S}} = \gamma_0 \frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} + \gamma_1^{\mathcal{S}} \left(\frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} \right)^2 + \gamma_2^{\mathcal{S}} \left(\frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} \right)^3 + \cdots$$

★ Integrating this equation, one gets

$$\Delta Z_{\mathcal{O}}^{\mathcal{S}}(\mu) = \left(2\beta_0 \frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} \right)^{-\frac{\gamma_0}{2\beta_0}} \exp\left\{ \int_0^{g^{\mathcal{S}}(\mu)} \mathrm{d}g' \left(\frac{\gamma^{\mathcal{S}}(g')}{\beta^{\mathcal{S}}(g')} + \frac{\gamma_0}{\beta_0\, g'} \right) \right\}$$

If one uses the RI scheme, β_2 is the same as $\overline{\mathrm{MS}}$

★ Therefore, all the scale and scheme dependence in Z-factor is captures in ΔZ. This allows to define the RGI (renormalization group invariant) operator, which is independent of scale and scheme.

$$Z_{\mathcal{O}}^{\mathrm{RGI}} = Z_{\mathcal{O}}^{\mathrm{RI}'}(\mu)\, \Delta Z_{\mathcal{O}}^{\mathrm{RI}'}(\mu) = Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV})\, \Delta Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV})$$

Exercise: Calculation of Z_q

Non-perturbatively in RI scheme

★ The scale dependence of Z-factors of operators is determined by its anomalous dimension

$$\gamma^{\mathcal{S}} = -\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \log Z_{\mathcal{S}} = \gamma_0 \frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} + \gamma_1^{\mathcal{S}} \left(\frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} \right)^2 + \gamma_2^{\mathcal{S}} \left(\frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} \right)^3 + \cdots$$

★ Integrating this equation, one gets

$$\Delta Z_{\mathcal{O}}^{\mathcal{S}}(\mu) = \left(2\beta_0 \frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} \right)^{-\frac{\gamma_0}{2\beta_0}} \exp\left\{ \int_0^{g^{\mathcal{S}}(\mu)} \mathrm{d}g' \left(\frac{\gamma^{\mathcal{S}}(g')}{\beta^{\mathcal{S}}(g')} + \frac{\gamma_0}{\beta_0\, g'} \right) \right\}$$

$$\beta^{\mathcal{S}} = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} g^{\mathcal{S}}(\mu) = -\beta_0 \frac{g^{\mathcal{S}}(\mu)^3}{16\pi^2} - \beta_1 \frac{g^{\mathcal{S}}(\mu)^5}{(16\pi^2)^2} - \beta_2^{\mathcal{S}} \frac{g^{\mathcal{S}}(\mu)^7}{(16\pi^2)^3} + \cdots$$

$$\begin{aligned} \beta_0 &= 11 - \frac{2}{3} N_f \,, \\ \beta_1 &= 102 - \frac{38}{3} N_f \,, \\ \beta_2 &= \frac{2857}{2} - \frac{5033}{18} N_f + \frac{325}{54} N_f^2 \end{aligned}$$

If one uses the RI scheme, β_2 is the same as $\overline{\mathrm{MS}}$

★ Therefore, all the scale and scheme dependence in Z-factor is captures in ΔZ. This allows to define the RGI (renormalization group invariant) operator, which is independent of scale and scheme.

$$Z_{\mathcal{O}}^{\mathrm{RGI}} = Z_{\mathcal{O}}^{\mathrm{RI}'}(\mu)\, \Delta Z_{\mathcal{O}}^{\mathrm{RI}'}(\mu) = Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV})\, \Delta Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV})$$

Renormalization of fermion field

★ Original RI-MOM scheme (perturbatively)

$$Z_q^{\mathrm{RI-MOM}} = -\frac{i}{48}\mathrm{Tr}\left[\gamma_\rho \frac{\partial S_q^{-1}(p)}{\partial p_\rho}\right]\Bigg|_{p^2=\mu^2}$$

★ RI-MOM not convenient in non-perturbative applications. More appropriate is the RI' scheme:

$$Z_q^{\mathrm{RI}'} = \frac{1}{12}\mathrm{Tr}\left[\frac{-i\sum_\rho \gamma_\rho \sin(p_\rho)}{\sum_\rho \sin(p_\rho)^2} S_q^{-1}(p)(p)\right]\Bigg|_{p^2=\mu^2}$$

★ The two schemes have different conversion functions to the $\overline{\mathrm{MS}}$ scheme

Contributions of Lattice Pert. Theory (2)

★ Renormalization of new operators for which non-perturbative prescriptions are not available

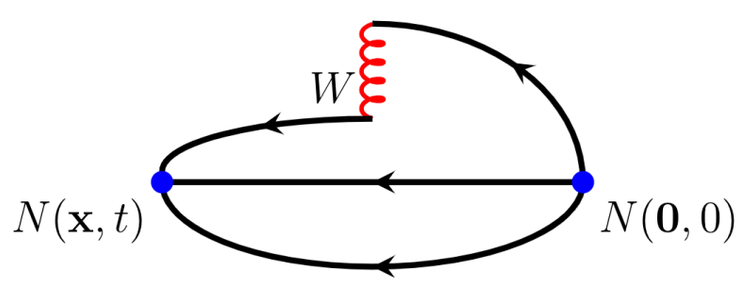
★ Example: non-local operators containing a Wilson line

$$\mathcal{O} \equiv \bar{\Psi}(z)\,\Gamma\,\mathscr{W}(z,0)\Psi(0)$$

★ Such matrix elements are linked to distribution functions (PDFs, GPDs, TMDs) that characterize the structure of the hadron (quasi-distributions approach)

[X. Ji, Phys. Rev. Lett. 110 (2013) 262002]
[X. Ji, Sci. China Phys. M.A. 57 (2014) 1407]

Diagrammatic representation of nucleon 3pt-function (connected) of non-local operator containing a Wilson line of length z

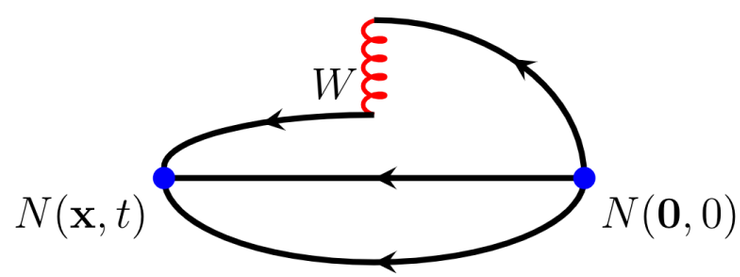


★ Prior 2017 lattice calculations missing a main ingredient:
– Renormalization
Thus, comparison with phenomenological data on PDFs is impossible!

Contributions of Lattice Pert. Theory (2)

Challenges of renormalization prescription development

★ Non-locality of operator



★ Power divergence in lattice regularization

[Dotsenko & Vergeles, Nucl. Phys. B169 (1980) 527]

$$e^{-c\frac{|z|}{a}}, \qquad c \sim 1$$

★ Renormalizability not proven

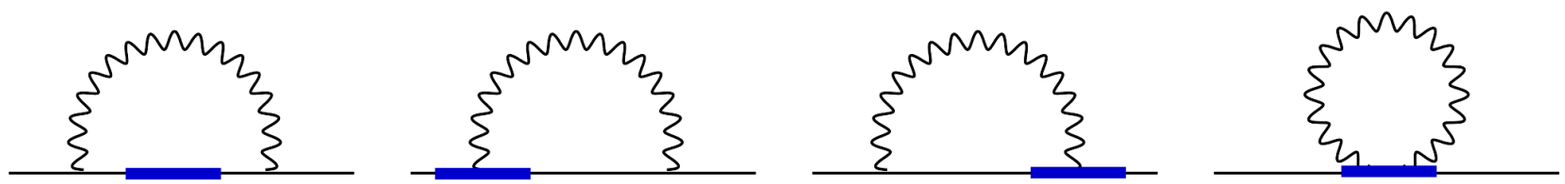
Progress in 2017:

- Diagrammatic expansion method [T. Ishikawa et al., PR D 96, 9 (2017) 094019, arXiv:1707.03107]
- Auxiliary heavy quark effective formalism [X. Ji et al., PRL 120 11 (2018) 112001, arXiv:1706.08962]

Contributions of Lattice Pert. Theory (2)

★ Solution: Lattice perturbation theory

Feynman diagrams



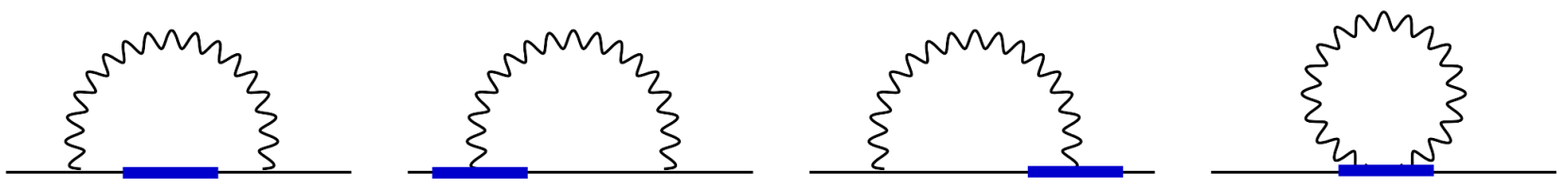
Strategy:

1. Perform the calculation in Dimensional Regularization (DR)
2. Isolate the poles and extract $Z^{DR,\overline{\mathrm{MS}}}$
3. Perform the calculation in Lattice Regularization (LR)
4. Extract $Z^{LR,\overline{\mathrm{MS}}}$ using the difference between DR and LR

Contributions of Lattice Pert. Theory (2)

★ One-loop calculation of non-local operators in LPT

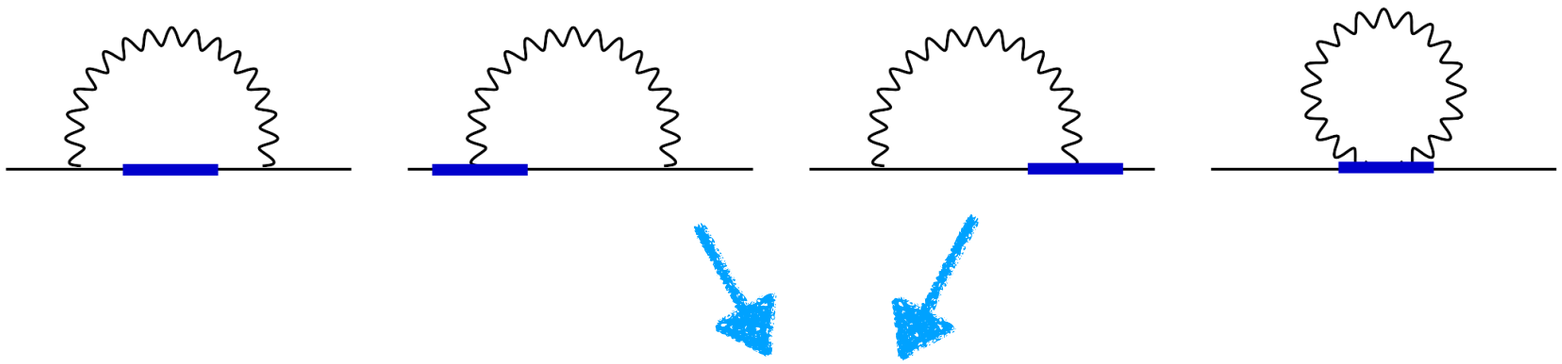
Feynman diagrams



Contributions of Lattice Pert. Theory (2)

★ One-loop calculation of non-local operators in LPT

Feynman diagrams



Most divergent contribution

$$I_{lat} = \int \frac{dp^4}{(2\pi)^4} \frac{e^{-i\,n\,p_\mu} - 1}{\sin\left(\frac{p_\mu}{2}\right)} \frac{\sin\left(p_\rho + a\,q_\rho\right)}{\widehat{p}^2\,\widehat{(p + a\,q)}^2}$$

$$\Big\downarrow \quad I_{lat} - I_1 : \text{Factorizable integral} \Rightarrow \text{Explicit extraction of } a\text{−dependence}$$

$$I_1 = \int \frac{dp^4}{(2\pi)^4} \frac{e^{-i\,n\,p_\mu} - 1}{\sin\left(\frac{p_\mu}{2}\right)} \left(\frac{\sin\left(p_\rho + a\,q_\rho\right)}{\widehat{p}^2\,\widehat{(p + a\,q)}^2} - \frac{\sin\left(\overline{p}_\rho + a\,q_\rho\right)}{\widehat{\overline{p}}^2\,\widehat{(\overline{p} + a\,q)}^2} \right)$$

$$\Big\downarrow \quad I_1 - I_2 : \text{Naive } a \to 0 \text{ limit} : e^{-i\,n\,p_\mu} \to 0,\ a\,q \to 0 \Rightarrow \text{ Constant integral}$$

$$I_2 = \int \frac{dp^4}{(2\pi)^4} \frac{e^{-i\,n\,p_\mu} - 1}{p_\mu/2} \left(\frac{p_\rho + a\,q_\rho}{p^2\,(p + a\,q)^2} - \frac{\overline{p}_\rho + a\,q_\rho}{\overline{p}^2\,(\overline{p} + a\,q)^2} \right)$$

$$\Big\downarrow \quad I_2 - I_3 : \text{Naive } a \to 0 \text{ limit} : e^{-i\,n\,p_\mu} \to 0,\ a\,q \to 0 \Rightarrow r\text{−dependent integral}$$

$$I_3 = \int_{p \le r} \frac{dp^4}{(2\pi)^4} \frac{e^{-i\,n\,p_\mu} - 1}{p_\mu/2} \left(\frac{(p_\rho + a\,q_\rho)}{p^2\,(p + a\,q)^2} - \frac{(\overline{p}_\rho + a\,q_\rho)}{\overline{p}^2\,(\overline{p} + a\,q)^2} \right)$$

$$= \int_{p \le r/a} \frac{dp^4}{(2\pi)^4} \frac{e^{-i\,z\,p_\mu} - 1}{p_\mu/2} \left(\frac{(p_\rho + q_\rho)}{p^2\,(p + q)^2} - \frac{(\overline{p}_\rho + q_\rho)}{\overline{p}^2\,(\overline{p} + q)^2} \right).$$

The extraction of IR divergences is much more complex than in DR.

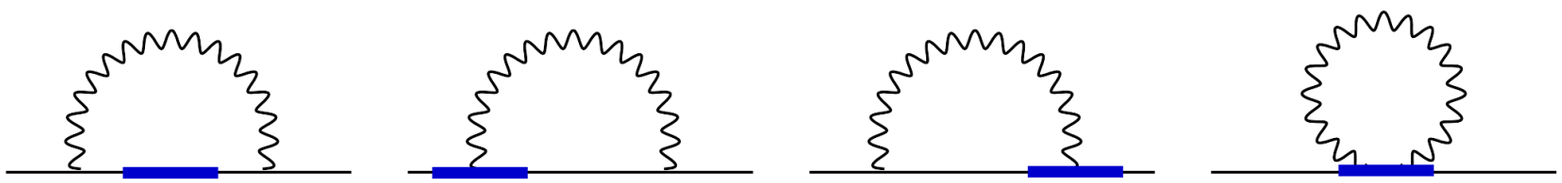
A generalization of the standard procedure of Kawai et al., in which one subtracts and adds to the original integrand its naïve Taylor expansion.

Here shown in Feynman gauge

Contributions of Lattice Pert. Theory (2)

★ One-loop calculation of non-local operators in LPT

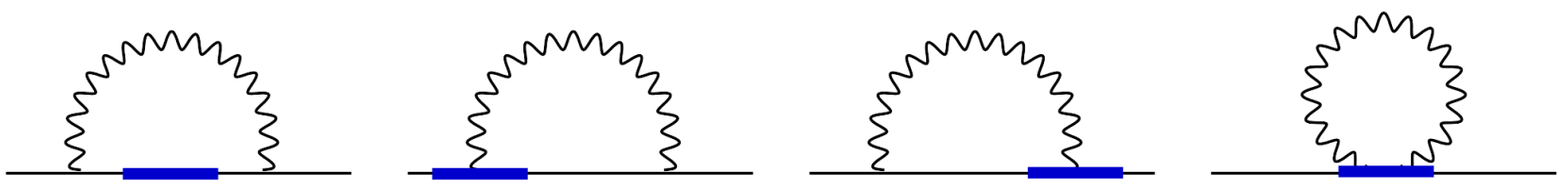
Feynman diagrams



Contributions of Lattice Pert. Theory (2)

★ One-loop calculation of non-local operators in LPT

Feynman diagrams



Vector operator

$$\Lambda_{V_\mu}^{1-\text{loop}} = \Lambda_{V_\mu}^{\text{tree}}\Bigg[1 + \frac{g^2\,C_f}{16\,\pi^2}\Bigg(8 + 4\gamma_E - \log(16) - 4F_2 + 4q|z|\,(F_4 - F_5) + 3\log\left(\frac{\bar{\mu}^2}{q^2}\right) + (\beta+2)\log(q^2 z^2)$$

$$+\beta\Big[-3 + 2\gamma_E - \log(4) + 2F_1 + 2(q^2 + 2{q_\mu}^2)G_3 - \frac{q^2 z^2}{2}(F_1 - F_2)\Big]$$

$$+4iq_\mu(-2G_1 + G_2 + z(F_3 - F_1 + F_2))$$

$$+i\beta q_\mu\Big[4G_1 - 4G_2 + 2z(F_2 - F_1) + 2q(-2G_4 + 3G_5)\Big]\Bigg)\Bigg]$$

$$+q\,e^{iq_\mu z}\,\frac{g^2\,C_f}{16\,\pi^2}\Bigg[-\frac{4q_\mu|z|}{q}F_5 + \beta q_\mu\Big[z^2(F_1 - F_2 - F_3) - 2G_3 + \frac{2|z|}{q}F_4\Big]$$

$$+4i(G_1 - G_2 + z(F_3 - F_1 + F_2))$$

$$+i\beta\Big[2(G_2 - G_1) + 2z(F_1 - F_2) + q(2G_4 - 2G_5 - z|z|F_5)\Big]\Bigg]$$

$$F_1(q,z) = \int_0^1 dx\, e^{-iq_\mu xz}\, K_0\left(q\,|z|\,\sqrt{(1-x)x}\right)$$

$$F_2(q,z) = \int_0^1 dx\, e^{-iq_\mu xz}\, x\, K_0\left(q\,|z|\,\sqrt{(1-x)x}\right)$$

$$F_3(q,z) = \int_0^1 dx\, e^{-iq_\mu xz}\, (1-x)^2\, K_0\left(q\,|z|\,\sqrt{(1-x)x}\right)$$

$$F_4(q,z) = \int_0^1 dx\, e^{-iq_\mu xz}\, \sqrt{(1-x)x}\, K_1\left(q\,|z|\,\sqrt{(1-x)x}\right)$$

$$F_5(q,z) = \int_0^1 dx\, e^{-iq_\mu xz}\, (1-x)\sqrt{(1-x)x}\, K_1\left(q\,|z|\,\sqrt{(1-x)x}\right)$$

$$G_1(q,z) = \int_0^1 dx \int_0^z d\zeta\, e^{-iq_\mu x\zeta}\, K_0\left(q\,|\zeta|\,\sqrt{(1-x)x}\right)$$

$$G_2(q,z) = \int_0^1 dx \int_0^z d\zeta\, e^{-iq_\mu x\zeta}\, x\, K_0\left(q\,|\zeta|\,\sqrt{(1-x)x}\right)$$

$$G_3(q,z) = \int_0^1 dx \int_0^z d\zeta\, e^{-iq_\mu x\zeta}\, \zeta\, x\,(1-x)\, K_0\left(q\,|\zeta|\,\sqrt{(1-x)x}\right)$$

$$G_4(q,z) = \int_0^1 dx \int_0^z d\zeta\, e^{-iq_\mu x\zeta}\, |\zeta|\,\sqrt{(1-x)x}\, K_1\left(q\,|\zeta|\,\sqrt{(1-x)x}\right)$$

$$G_5(q,z) = \int_0^1 dx \int_0^z d\zeta\, e^{-iq_\mu x\zeta}\, |\zeta|\, x\sqrt{(1-x)x}\, K_1\left(q\,|\zeta|\,\sqrt{(1-x)x}\right)$$

Contributions of Lattice Pert. Theory (3)

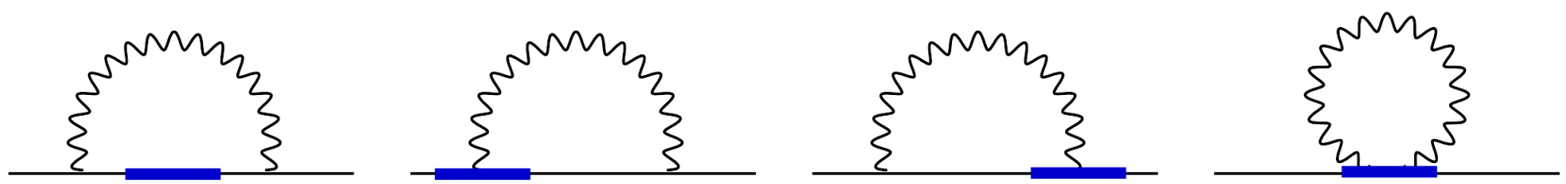
★ Qualitative understanding of renormalization pattern

Contributions of Lattice Pert. Theory (3)

★ Qualitative understanding of renormalization pattern

Example: Non-local operators … again

Feynman diagrams

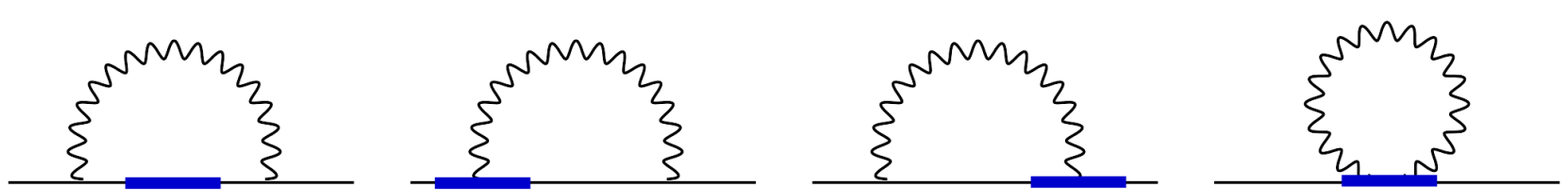


Contributions of Lattice Pert. Theory (3)

★ Qualitative understanding of renormalization pattern

Example: Non-local operators … again

Feynman diagrams



Final results

$$\langle \psi\, \mathcal{O}_\Gamma\, \bar{\psi} \rangle^{DR,\,\overline{\rm MS}}_{\rm amp} - \langle \psi\, \mathcal{O}_\Gamma\, \bar{\psi} \rangle^{LR}_{\rm amp} = \frac{g^2\, C_f}{16\,\pi^2}\, e^{i\, q_\mu\, z} \times \mathcal{F}$$

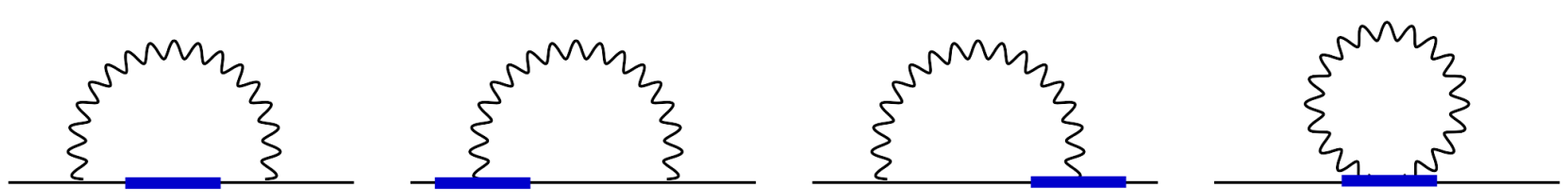
$$\mathcal{F} = \Gamma \left(c_1 + c_2\, \beta + c_3\, \frac{|z|}{a} + \log\left(a^2 \bar{\mu}^2\right) (4-\beta) \right) + (\Gamma \cdot \gamma_\mu + \gamma_\mu \cdot \Gamma) \left(c_4 + c_5\, c_{\rm SW} \right)$$

Contributions of Lattice Pert. Theory (3)

★ Qualitative understanding of renormalization pattern

Example: Non-local operators … again

Feynman diagrams



Final results

$$\langle \psi\, \mathcal{O}_\Gamma\, \bar{\psi} \rangle^{DR,\,\overline{\rm MS}}_{\rm amp} - \langle \psi\, \mathcal{O}_\Gamma\, \bar{\psi} \rangle^{LR}_{\rm amp} = \frac{g^2\, C_f}{16\,\pi^2}\, e^{i\, q_\mu\, z} \times \mathcal{F}$$

$$\mathcal{F} = \Gamma \left(c_1 + c_2\, \beta + c_3\, \frac{|z|}{a} + \log\left(a^2 \bar{\mu}^2\right) (4-\beta) \right) + (\Gamma \cdot \gamma_\mu + \gamma_\mu \cdot \Gamma) \left(c_4 + c_5\, c_{\rm SW} \right)$$

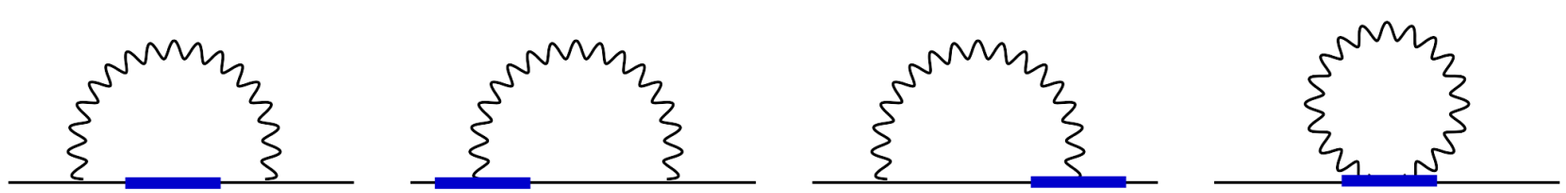
linear divergence

Contributions of Lattice Pert. Theory (3)

★ Qualitative understanding of renormalization pattern

Example: Non-local operators … again

Feynman diagrams



Final results

$$\langle \psi\, \mathcal{O}_\Gamma\, \bar{\psi} \rangle^{DR,\,\overline{\rm MS}}_{\rm amp} - \langle \psi\, \mathcal{O}_\Gamma\, \bar{\psi} \rangle^{LR}_{\rm amp} = \frac{g^2\, C_f}{16\,\pi^2}\, e^{i\, q_\mu\, z} \times \mathcal{F}$$

$$\mathcal{F} = \Gamma \left(c_1 + c_2\, \beta + c_3\, \frac{|z|}{a} + \log\left(a^2 \bar{\mu}^2\right) (4-\beta) \right) + (\Gamma \cdot \gamma_\mu + \gamma_\mu \cdot \Gamma) \left(c_4 + c_5\, c_{\rm SW} \right)$$

linear divergence

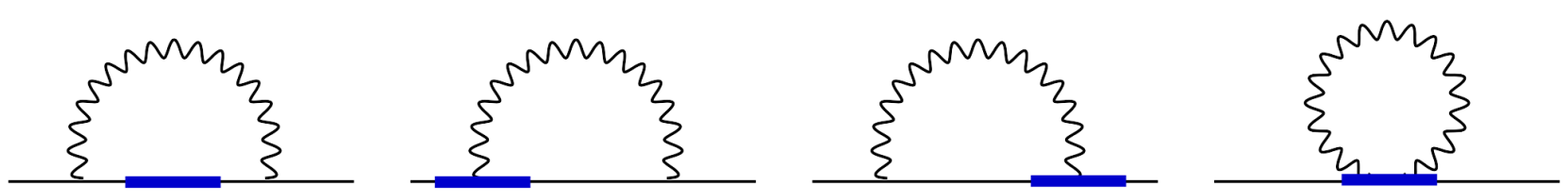
mixing term

Contributions of Lattice Pert. Theory (3)

★ Qualitative understanding of renormalization pattern

Example: Non-local operators … again

Feynman diagrams



Final results

$$\langle \psi\, \mathcal{O}_\Gamma\, \bar{\psi} \rangle^{DR,\,\overline{\rm MS}}_{\rm amp} - \langle \psi\, \mathcal{O}_\Gamma\, \bar{\psi} \rangle^{LR}_{\rm amp} = \frac{g^2\, C_f}{16\,\pi^2}\, e^{i\, q_\mu z} \times \mathcal{F}$$

$$\mathcal{F} = \Gamma \left(c_1 + c_2\, \beta + c_3\, \frac{|z|}{a} + \log\left(a^2 \bar{\mu}^2\right) (4-\beta) \right) + (\Gamma \cdot \gamma_\mu + \gamma_\mu \cdot \Gamma) \left(c_4 + c_5\, c_{\rm SW} \right)$$

linear divergence

mixing term

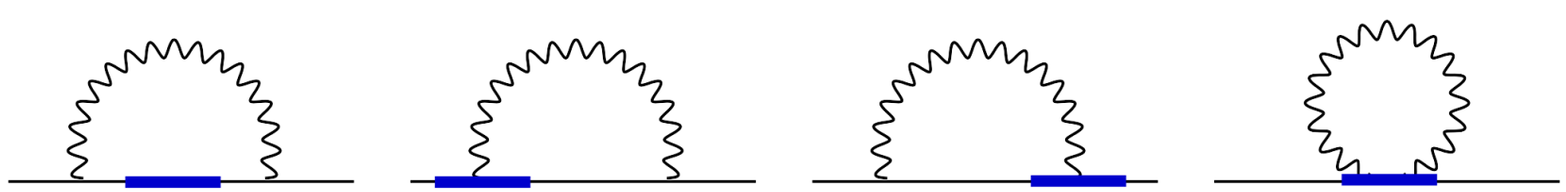
Unexpected feature!

Contributions of Lattice Pert. Theory (3)

★ Qualitative understanding of renormalization pattern

Example: Non-local operators … again

Feynman diagrams



Final results

$$\langle \psi\, \mathcal{O}_\Gamma\, \bar{\psi} \rangle^{DR,\,\overline{\rm MS}}_{\rm amp} - \langle \psi\, \mathcal{O}_\Gamma\, \bar{\psi} \rangle^{LR}_{\rm amp} = \frac{g^2\, C_f}{16\,\pi^2}\, e^{i\, q_\mu z} \times \mathcal{F}$$

$$\mathcal{F} = \Gamma\Big(c_1 + c_2\,\beta + c_3\,\frac{|z|}{a} + \log\left(a^2\bar{\mu}^2\right)(4-\beta)\Big) + (\Gamma\cdot\gamma_\mu + \gamma_\mu\cdot\Gamma)\,\Big(c_4 + c_5\, c_{\rm SW}\Big)$$

linear divergence

mixing term

Unexpected feature!

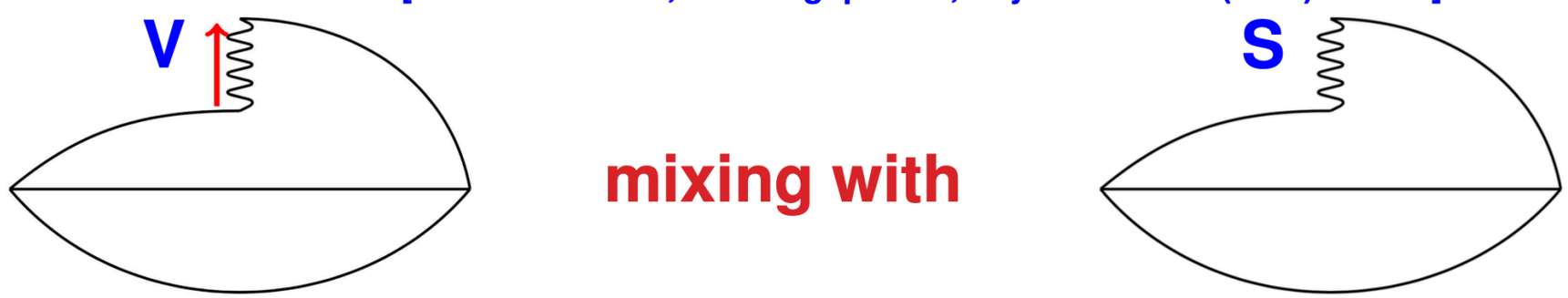
Mixing depends on Dirac structure (Γ) of operators.

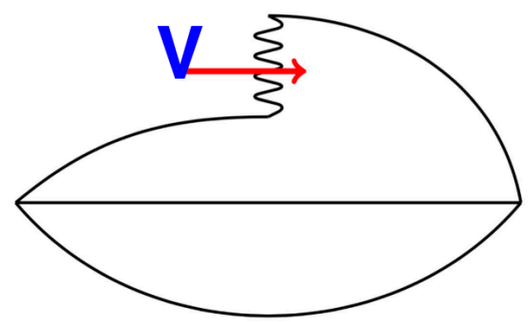
If $\{\Gamma, \gamma_\mu\} \neq 0$ mixing occurs

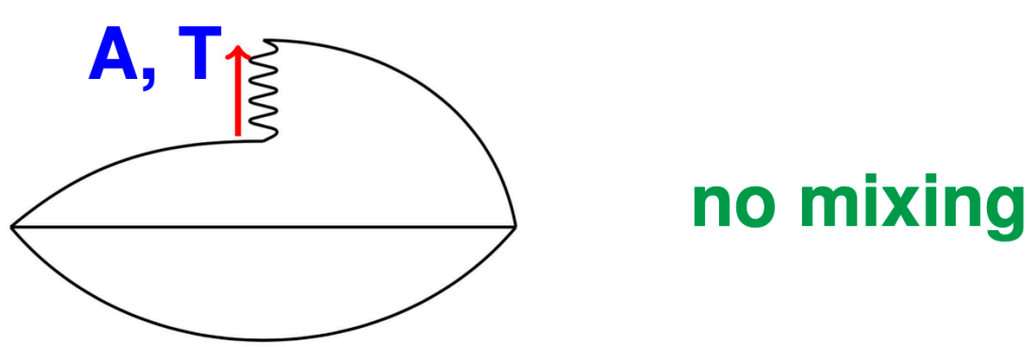
Contributions of Lattice Pert. Theory (3)

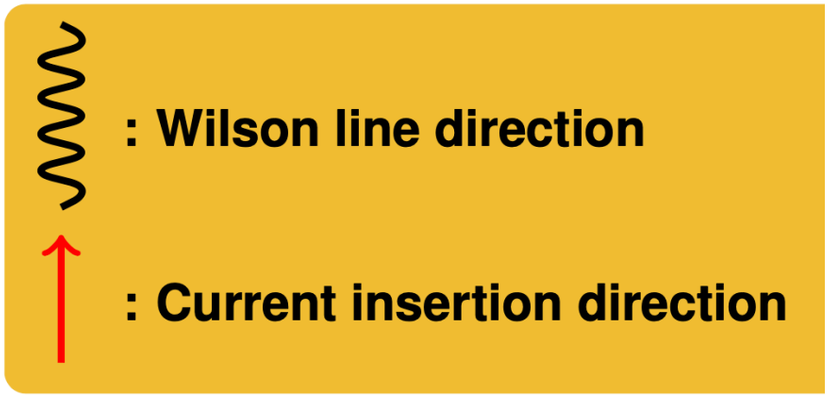
Mixing depends on the relation between the current & Wilson line direction

[M. Constantinou, H. Panagopoulos, Phys. Rev. D 96 (2017) 054506]









Contributions of Lattice Pert. Theory (3)

Benefits of perturbative calculation:

[M. Constantinou, H. Panagopoulos, PRD 96 (2017) 054506, [arXiv:1705.11193]]

1. Dirac structures that avoid mixing have been implemented (e.g., vector operator for the unpolarized PDF)

2. Understanding the mixing pattern helps develop non-perturbative renormalization prescription

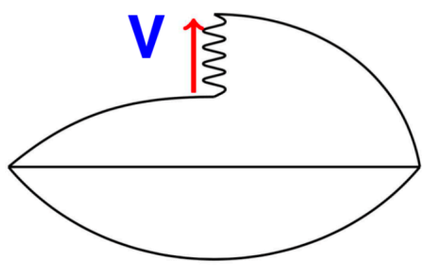
[C. Alexandrou, et al., Nucl. Phys. B 923 (2017) 394 (Frontier Article), [arXiv:1706.00265]]

Contributions of Lattice Pert. Theory (3)

Benefits of perturbative calculation:

[M. Constantinou, H. Panagopoulos, PRD 96 (2017) 054506, [arXiv:1705.11193]]

1. Dirac structures that avoid mixing have been implemented (e.g., vector operator for the unpolarized PDF)



Used until 2017

2. Understanding the mixing pattern helps develop non-perturbative renormalization prescription

[C. Alexandrou, et al., Nucl. Phys. B 923 (2017) 394 (Frontier Article), [arXiv:1706.00265]]

Contributions of Lattice Pert. Theory (3)

Benefits of perturbative calculation:

[M. Constantinou, H. Panagopoulos, PRD 96 (2017) 054506, [arXiv:1705.11193]]

1. Dirac structures that avoid mixing have been implemented (e.g., vector operator for the unpolarized PDF)

2. Understanding the mixing pattern helps develop non-perturbative renormalization prescription

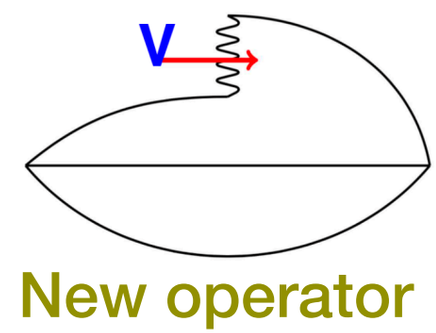
[C. Alexandrou, et al., Nucl. Phys. B 923 (2017) 394 (Frontier Article), [arXiv:1706.00265]]

Contributions of Lattice Pert. Theory (3)

Benefits of perturbative calculation:

[M. Constantinou, H. Panagopoulos, PRD 96 (2017) 054506, [arXiv:1705.11193]]

1. Dirac structures that avoid mixing have been implemented (e.g., vector operator for the unpolarized PDF)



2. Understanding the mixing pattern helps develop non-perturbative renormalization prescription

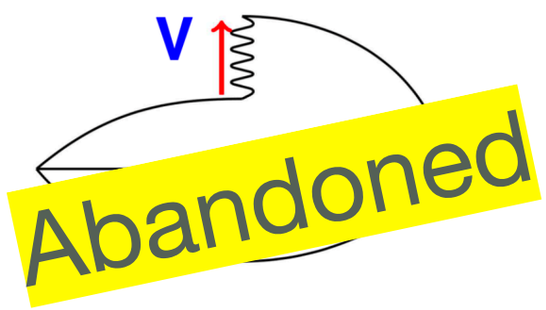
[C. Alexandrou, et al., Nucl. Phys. B 923 (2017) 394 (Frontier Article), [arXiv:1706.00265]]

Contributions of Lattice Pert. Theory (3)

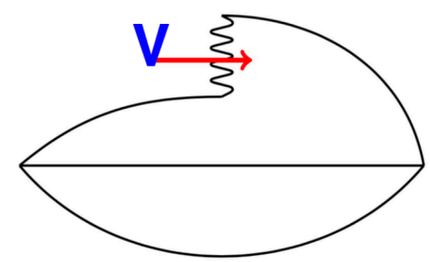
Benefits of perturbative calculation:

[M. Constantinou, H. Panagopoulos, PRD 96 (2017) 054506, [arXiv:1705.11193]]

1. Dirac structures that avoid mixing have been implemented (e.g., vector operator for the unpolarized PDF)



Used until 2017



New operator

2. Understanding the mixing pattern helps develop non-perturbative renormalization prescription

[C. Alexandrou, et al., Nucl. Phys. B 923 (2017) 394 (Frontier Article), [arXiv:1706.00265]]

Absence of mixing

$$Z_{\mathcal{O}}(z) = \frac{Z_q}{\mathcal{V}_{\mathcal{O}}(z)}\,,$$

$$\mathcal{V}_{\mathcal{O}} = \frac{\mathrm{Tr}}{12}\left[\mathcal{V}(p)\left(\mathcal{V}^{\mathrm{Born}}(p)\right)^{-1}\right]\Big|_{p=\bar{\mu}}$$

Presence of mixing

$$\begin{pmatrix}\mathcal{O}_V^R(P_3,z)\\ \mathcal{O}_S^R(P_3,z)\end{pmatrix} = \hat{Z}(z)\cdot\begin{pmatrix}\mathcal{O}_V(P_3,z)\\ \mathcal{O}_S(P_3,z)\end{pmatrix}, \qquad Z_q^{-1}\,\hat{Z}(z)\,\hat{\mathcal{V}}(p,z)\Big|_{p=\bar{\mu}} = \hat{1}$$

$$h_V^R(P_3,z) = Z_{VV}(z)\;h_V(P_3,z) + Z_{VS}(z)\;h_S(P_3,z)$$

Contributions of Lattice Pert. Theory (4)

Operator improvement

- ★ Symanzik improvement:
  A systematic method to remove discretization effects order by order in a using counter-terms [K. Symanzik, Nucl. Phys. B226 (1983) 187]

- ★ Further development by Luscher & Weisz (on-shell quantities)
  [M. Luscher and P. Weisz, Comm. Math. Phys. 97 (1985) 59]

- ★ Development of basis of improved operators (off-shell improvement), e.g.,

$$\left(\mathcal{O}^S\right)^{\text{imp}} = \left(\bar{\psi}\psi\right)^{\text{imp}} = (1 + a\,m\,c_0)\bar{\psi}\psi - \frac{1}{2}ac_1\bar{\psi}\,\overleftrightarrow{\not{D}}\psi$$

- ★ Improvement coefficients (e.g., c_0, c_1) calculated in LPT
  [S. Aoki et al., Phys. Rev. D58 (1998) 074505 [hep-lat/9802034]]
  [S. Capitani et al., Nucl. Phys. B593 (2001) 183 [hep-lat/0007004]]

Contributions of Lattice Pert. Theory (5)

- ★ First (analytic) calculations to $\mathcal{O}(g^2 a)$
  (focus on operator improvement)

[S. Capitani et al., Nucl. Phys. B593 (2001) 183 [hep-lat/0007004]]
[S. Aoki et al., Phys. Rev. D58 (1998) 074505 [hep-lat/9802034]]

Contributions of Lattice Pert. Theory (5)

★ First (analytic) calculations to $\mathcal{O}(g^2 a)$
(focus on operator improvement)

[S. Capitani et al., Nucl. Phys. B593 (2001) 183 [hep-lat/0007004]]
[S. Aoki et al., Phys. Rev. D58 (1998) 074505 [hep-lat/9802034]]

★ Another idea on utilizing calculations up to $\mathcal{O}(g^2 a)$:
calculate analytically contributions beyond leading order in a, and then subtract them from non-perturbative estimates

[M. Constantinou et al., *JHEP* 10 (2009) 064, [arXiv:0907.0381]]
[M. Constantinou et al. (ETMC), *JHEP* 08 (2010) 068 [arXiv:1004.1115]]
[C. Alexandrou et al. (ETMC), PRD 83 (2011) 014503[arXiv:1006.1920]]
[M. Constantinou et al., PRD 83 (2011) 014503 [arXiv:1011.6059]]
[C. Alexandrou et al. (ETMC), *PRD* 86 (2012) 014505 [arXiv:1201.5025]]

Contributions of Lattice Pert. Theory (5)

★ First (analytic) calculations to $\mathcal{O}(g^2 a)$
(focus on operator improvement)

[S. Capitani et al., Nucl. Phys. B593 (2001) 183 [hep-lat/0007004]]
[S. Aoki et al., Phys. Rev. D58 (1998) 074505 [hep-lat/9802034]]

★ Another idea on utilizing calculations up to $\mathcal{O}(g^2 a)$:
calculate analytically contributions beyond leading order in a, and then subtract them from non-perturbative estimates

pert. calculation → [M. Constantinou et al., *JHEP* 10 (2009) 064, [arXiv:0907.0381]]
Implementation → [M. Constantinou et al. (ETMC), *JHEP* 08 (2010) 068 [arXiv:1004.1115]]
[C. Alexandrou et al. (ETMC), PRD 83 (2011) 014503[arXiv:1006.1920]]
[M. Constantinou et al., PRD 83 (2011) 014503 [arXiv:1011.6059]]
[C. Alexandrou et al. (ETMC), *PRD* 86 (2012) 014505 [arXiv:1201.5025]]

Contributions of Lattice Pert. Theory (5)

★ First (analytic) calculations to $\mathcal{O}(g^2 a)$
(focus on operator improvement)

[S. Capitani et al., Nucl. Phys. B593 (2001) 183 [hep-lat/0007004]]
[S. Aoki et al., Phys. Rev. D58 (1998) 074505 [hep-lat/9802034]]

★ Another idea on utilizing calculations up to $\mathcal{O}(g^2 a)$:
calculate analytically contributions beyond leading order in a, and then subtract them from non-perturbative estimates

pert. calculation → [M. Constantinou et al., *JHEP* 10 (2009) 064, [arXiv:0907.0381]]
Implementation → [M. Constantinou et al. (ETMC), *JHEP* 08 (2010) 068 [arXiv:1004.1115]]
[C. Alexandrou et al. (ETMC), PRD 83 (2011) 014503[arXiv:1006.1920]]
[M. Constantinou et al., PRD 83 (2011) 014503 [arXiv:1011.6059]]
[C. Alexandrou et al. (ETMC), *PRD* 86 (2012) 014505 [arXiv:1201.5025]]

★ Another idea on utilizing calculations up to $\mathcal{O}(g^2 a^\infty)$:
calculate numerically the Green functions at all order in a, and then subtract them from non-perturbative estimates

[M. Constantinou et al. (QCDSF), PRD 87 (2013) 9, 096019, [arXiv:1303.6776]]
[M. Constantinou et al. (QCDSF), PRD 91 (2015) 1, 014502, [arXiv:1408.6047]]
[C. Alexandrou et al. (ETMC), PRD 95 (2017) 3, 034505, [arXiv:1509.00213]]
[C. Alexandrou et al. (ETMC), arXiv:2104.13408]

Contributions of Lattice Pert. Theory (5)

★ First (analytic) calculations to $\mathcal{O}(g^2 a)$
(focus on operator improvement)

[S. Capitani et al., Nucl. Phys. B593 (2001) 183 [hep-lat/0007004]]
[S. Aoki et al., Phys. Rev. D58 (1998) 074505 [hep-lat/9802034]]

★ Another idea on utilizing calculations up to $\mathcal{O}(g^2 a)$:
calculate analytically contributions beyond leading order in a, and then subtract them from non-perturbative estimates

pert. calculation → [M. Constantinou et al., *JHEP* 10 (2009) 064, [arXiv:0907.0381]]
Implementation → [M. Constantinou et al. (ETMC), *JHEP* 08 (2010) 068 [arXiv:1004.1115]]
[C. Alexandrou et al. (ETMC), PRD 83 (2011) 014503[arXiv:1006.1920]]
[M. Constantinou et al., PRD 83 (2011) 014503 [arXiv:1011.6059]]
[C. Alexandrou et al. (ETMC), *PRD* 86 (2012) 014505 [arXiv:1201.5025]]

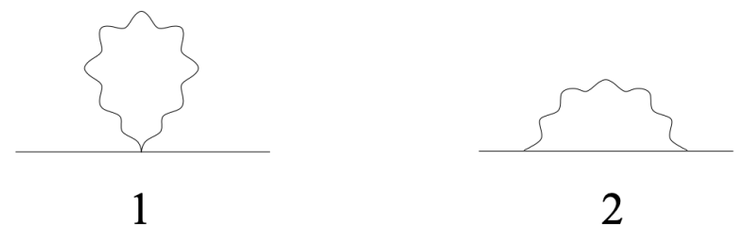
★ Another idea on utilizing calculations up to $\mathcal{O}(g^2 a^\infty)$:
calculate numerically the Green functions at all order in a, and then subtract them from non-perturbative estimates

[M. Constantinou et al. (QCDSF), PRD 87 (2013) 9, 096019, [arXiv:1303.6776]]
[M. Constantinou et al. (QCDSF), PRD 91 (2015) 1, 014502, [arXiv:1408.6047]]
[C. Alexandrou et al. (ETMC), PRD 95 (2017) 3, 034505, [arXiv:1509.00213]]
[C. Alexandrou et al. (ETMC), arXiv:2104.13408]

Advantage: Synergy between perturbative and non-perturbative results improves final estimates for renormalization functions

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a)$ in Z_q



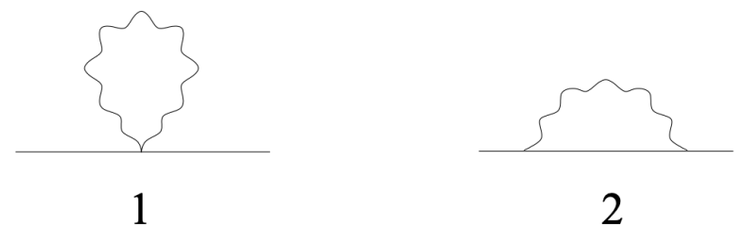
1-loop diagrams for the fermion propagator

★ Inverse propagator up to $\mathcal{O}(g^2 a)$ (massless case):

$$\sin(ap_\nu) = ap_\nu + \mathcal{O}(a^3)$$
$$\cos(ap_\nu) = \delta_{\nu\nu} - \frac{1}{2}(ap_\nu)^2 + \mathcal{O}(a^4)$$

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a)$ in Z_q



1-loop diagrams for the fermion propagator

★ Inverse propagator up to $\mathcal{O}(g^2 a)$ (massless case):

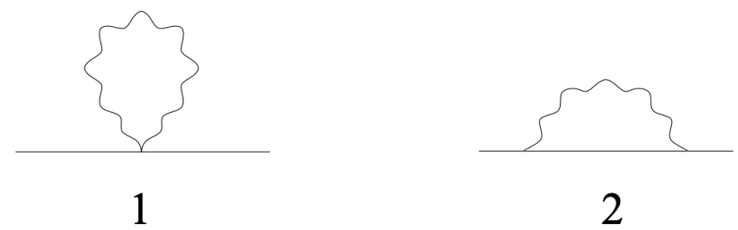
$$\sin(ap_\nu) = ap_\nu + \mathcal{O}(a^3)$$
$$\cos(ap_\nu) = \delta_{\nu\nu} - \frac{1}{2}(ap_\nu)^2 + \mathcal{O}(a^4)$$

$$\begin{aligned}
S^{-1}_{(p)} \;=\;& i\,\not{p} + \frac{a}{2}p^2 - i\frac{a^2}{6}\,\not{p}^3 \\
&- i\,\not{p}\,\tilde{g}^2\Big[\varepsilon^{(0,1)} - 4.79200956(5)\,\lambda + \varepsilon^{(0,2)}\,c_{\rm SW} + \varepsilon^{(0,3)}\,c_{\rm SW}^2 + \lambda\ln(a^2p^2)\Big] \\
&- a\,p^2\,\tilde{g}^2\Big[\varepsilon^{(1,1)} - 3.86388443(2)\,\lambda + \varepsilon^{(1,2)}\,c_{\rm SW} + \varepsilon^{(1,3)}\,c_{\rm SW}^2 - \frac{1}{2}\,(3 - 2\,\lambda - 3\,c_{\rm SW})\,\ln(a^2p^2)\Big] \\
&- i\,a^2\,\not{p}^3\,\tilde{g}^2\Big[\varepsilon^{(2,1)} + 0.507001567(9)\,\lambda + \varepsilon^{(2,2)}\,c_{\rm SW} + \varepsilon^{(2,3)}\,c_{\rm SW}^2 \\
&\qquad + \left(\frac{101}{120} - \frac{11}{30}C_2 - \frac{\lambda}{6}\right)\,\ln(a^2p^2)\Big] \\
&- i\,a^2\,p^2\,\not{p}\,\tilde{g}^2\Big[\varepsilon^{(2,4)} + 1.51604667(9)\,\lambda + \varepsilon^{(2,5)}\,c_{\rm SW} + \varepsilon^{(2,6)}\,c_{\rm SW}^2 \\
&\qquad + \left(\frac{59}{240} + \frac{c_1}{2} + \frac{C_2}{60} - \frac{1}{4}\left(\frac{3}{2}\lambda + c_{\rm SW} + c_{\rm SW}^2\right)\right)\,\ln(a^2p^2)\Big] \\
&- i\,a^2\,\not{p}\,\frac{\sum_\mu p_\mu^4}{p^2}\,\tilde{g}^2\Big[-\frac{3}{80} - \frac{C_2}{10} - \frac{5}{48}\lambda\Big]
\end{aligned} \tag{24}$$

[M. Constantinou et al., *JHEP* 10 (2009) 064, [arXiv:0907.0381]]

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a)$ in Z_q



1-loop diagrams for the fermion propagator

★ Inverse propagator up to $\mathcal{O}(g^2 a)$ (massless case):

$$\sin(ap_\nu) = ap_\nu + \mathcal{O}(a^3)$$
$$\cos(ap_\nu) = \delta_{\nu\nu} - \frac{1}{2}(ap_\nu)^2 + \mathcal{O}(a^4)$$

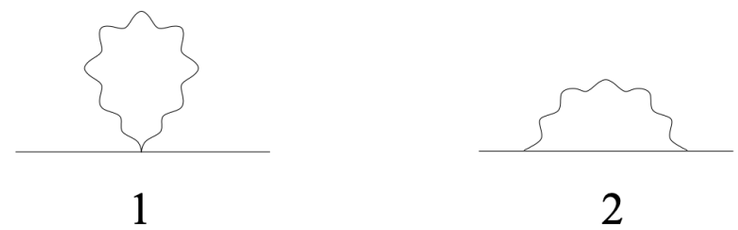
$$\begin{aligned}
S^{-1}_{(p)} &= i\,\not{p} + \frac{a}{2}p^2 - i\frac{a^2}{6}\not{p}^3 \\
&- i\,\not{p}\tilde{g}^2\Big[\varepsilon^{(0,1)} - 4.79200956(5)\,\lambda + \varepsilon^{(0,2)}\,c_{\rm SW} + \varepsilon^{(0,3)}\,c_{\rm SW}^2 + \lambda\ln(a^2p^2)\Big] \\
&- a\,p^2\,\tilde{g}^2\Big[\varepsilon^{(1,1)} - 3.86388443(2)\,\lambda + \varepsilon^{(1,2)}\,c_{\rm SW} + \varepsilon^{(1,3)}\,c_{\rm SW}^2 - \frac{1}{2}\,(3 - 2\,\lambda - 3\,c_{\rm SW})\,\ln(a^2p^2)\Big] \\
&- i\,a^2\,\not{p}^3\,\tilde{g}^2\Big[\varepsilon^{(2,1)} + 0.507001567(9)\,\lambda + \varepsilon^{(2,2)}\,c_{\rm SW} + \varepsilon^{(2,3)}\,c_{\rm SW}^2 \\
&\qquad + \left(\frac{101}{120} - \frac{11}{30}C_2 - \frac{\lambda}{6}\right)\,\ln(a^2p^2)\Big] \\
&- i\,a^2\,p^2\,\not{p}\,\tilde{g}^2\Big[\varepsilon^{(2,4)} + 1.51604667(9)\,\lambda + \varepsilon^{(2,5)}\,c_{\rm SW} + \varepsilon^{(2,6)}\,c_{\rm SW}^2 \\
&\qquad + \left(\frac{59}{240} + \frac{c_1}{2} + \frac{C_2}{60} - \frac{1}{4}\left(\frac{3}{2}\lambda + c_{\rm SW} + c_{\rm SW}^2\right)\right)\,\ln(a^2p^2)\Big] \\
&- i\,a^2\,\not{p}\,\frac{\sum_\mu p_\mu^4}{p^2}\,\tilde{g}^2\Big[-\frac{3}{80} - \frac{C_2}{10} - \frac{5}{48}\lambda\Big]
\end{aligned} \tag{24}$$

[M. Constantinou et al., *JHEP* 10 (2009) 064, [arXiv:0907.0381]]

★ Goal is NOT to provide Z_q from LPT. We want identify cutoff effects

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a)$ in Z_q



1-loop diagrams for the fermion propagator

★ Inverse propagator up to $\mathcal{O}(g^2 a)$ (massless case):

$$\sin(ap_\nu) = ap_\nu + \mathcal{O}(a^3)$$
$$\cos(ap_\nu) = \delta_{\nu\nu} - \frac{1}{2}(ap_\nu)^2 + \mathcal{O}(a^4)$$

$$\begin{aligned}
S^{-1}_{(p)} \;=\; & i\,\not{p} + \frac{a}{2}p^2 - i\frac{a^2}{6}\not{p}^3 \\
& - i\,\not{p}\,\tilde{g}^2\Big[\varepsilon^{(0,1)} - 4.79200956(5)\,\lambda + \varepsilon^{(0,2)}\,c_{\rm SW} + \varepsilon^{(0,3)}\,c_{\rm SW}^2 + \lambda\ln(a^2p^2)\Big] \\
& - a\,p^2\,\tilde{g}^2\Big[\varepsilon^{(1,1)} - 3.86388443(2)\,\lambda + \varepsilon^{(1,2)}\,c_{\rm SW} + \varepsilon^{(1,3)}\,c_{\rm SW}^2 - \frac{1}{2}\,(3 - 2\,\lambda - 3\,c_{\rm SW})\,\ln(a^2p^2)\Big] \\
& - i\,a^2\,\not{p}^3\,\tilde{g}^2\Big[\varepsilon^{(2,1)} + 0.507001567(9)\,\lambda + \varepsilon^{(2,2)}\,c_{\rm SW} + \varepsilon^{(2,3)}\,c_{\rm SW}^2 \\
& \qquad + \left(\frac{101}{120} - \frac{11}{30}C_2 - \frac{\lambda}{6}\right)\ln(a^2p^2)\Big] \\
& - i\,a^2\,p^2\,\not{p}\,\tilde{g}^2\Big[\varepsilon^{(2,4)} + 1.51604667(9)\,\lambda + \varepsilon^{(2,5)}\,c_{\rm SW} + \varepsilon^{(2,6)}\,c_{\rm SW}^2 \\
& \qquad + \left(\frac{59}{240} + \frac{c_1}{2} + \frac{C_2}{60} - \frac{1}{4}\left(\frac{3}{2}\lambda + c_{\rm SW} + c_{\rm SW}^2\right)\right)\ln(a^2p^2)\Big] \\
& - i\,a^2\,\not{p}\,\frac{\sum_\mu p_\mu^4}{p^2}\,\tilde{g}^2\Big[-\frac{3}{80} - \frac{C_2}{10} - \frac{5}{48}\lambda\Big] \qquad (24)
\end{aligned}$$

[M. Constantinou et al., *JHEP* 10 (2009) 064, [arXiv:0907.0381]]

★ Goal is NOT to provide Z_q from LPT. We want identify cutoff effects

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a)$ in Z_q

1 2

1-loop diagrams for the fermion propagator

★ Inverse propagator up to $\mathcal{O}(g^2 a)$ (massless case):

$$\sin(ap_\nu) = ap_\nu + \mathcal{O}(a^3)$$
$$\cos(ap_\nu) = \delta_{\nu\nu} - \frac{1}{2}(ap_\nu)^2 + \mathcal{O}(a^4)$$

$$\begin{aligned}
S^{-1}_{(p)} &= i\,\not{p} + \frac{a}{2}p^2 - i\frac{a^2}{6}\not{p}^3 \\
&- i\,\not{p}\,\tilde{g}^2\Big[\varepsilon^{(0,1)} - 4.79200956(5)\,\lambda + \varepsilon^{(0,2)}\,c_{\rm SW} + \varepsilon^{(0,3)}\,c_{\rm SW}^2 + \lambda\ln(a^2p^2)\Big] \\
&- a\,p^2\,\tilde{g}^2\Big[\varepsilon^{(1,1)} - 3.86388443(2)\,\lambda + \varepsilon^{(1,2)}\,c_{\rm SW} + \varepsilon^{(1,3)}\,c_{\rm SW}^2 - \frac{1}{2}(3 - 2\,\lambda - 3\,c_{\rm SW})\,\ln(a^2p^2)\Big] \\
&- i\,a^2\,\not{p}^3\,\tilde{g}^2\Big[\varepsilon^{(2,1)} + 0.507001567(9)\,\lambda + \varepsilon^{(2,2)}\,c_{\rm SW} + \varepsilon^{(2,3)}\,c_{\rm SW}^2 \\
&\qquad + \left(\frac{101}{120} - \frac{11}{30}C_2 - \frac{\lambda}{6}\right)\ln(a^2p^2)\Big] \\
&- i\,a^2\,p^2\,\not{p}\,\tilde{g}^2\Big[\varepsilon^{(2,4)} + 1.51604667(9)\,\lambda + \varepsilon^{(2,5)}\,c_{\rm SW} + \varepsilon^{(2,6)}\,c_{\rm SW}^2 \\
&\qquad + \left(\frac{59}{240} + \frac{c_1}{2} + \frac{C_2}{60} - \frac{1}{4}\left(\frac{3}{2}\lambda + c_{\rm SW} + c_{\rm SW}^2\right)\right)\ln(a^2p^2)\Big] \\
&- i\,a^2\,\not{p}\,\frac{\sum_\mu p_\mu^4}{p^2}\,\tilde{g}^2\Big[-\frac{3}{80} - \frac{C_2}{10} - \frac{5}{48}\lambda\Big]
\end{aligned} \tag{24}$$

Unwanted effects that are also present in the non-perturbative calculation

[M. Constantinou et al., *JHEP* 10 (2009) 064, [arXiv:0907.0381]]

★ Goal is NOT to provide Z_q from LPT. We want identify cutoff effects

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2a^2)$ in Z_q

★ Strategy:

- Calculate numerically the inverse propagator the values of the action parameters and momentum used in non-perturbative estimate

- Calculate Z_q non-perturbatively

- Subtract the artifacts from the estimates

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2a^2)$ in Z_q

★ Strategy:

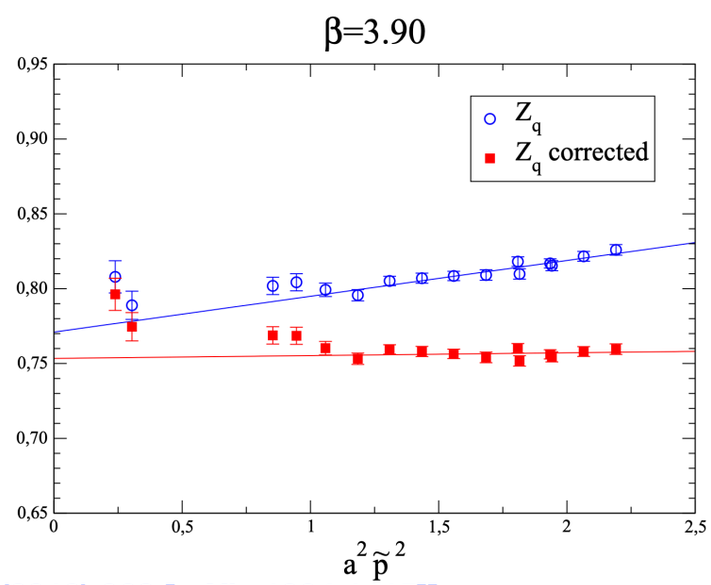
- Calculate numerically the inverse propagator the values of the action parameters and momentum used in non-perturbative estimate

- Calculate Z_q non-perturbatively

- Subtract the artifacts from the estimates

Application for twisted mass fermions and tree-level Symanzik improved gluons

β	a (fm)	$a\mu_0$	m_π (GeV)	$L^3 \times T$
3.9	0.089	0.0040	0.3021(14)	$24^3 \times 48$
3.9	0.089	0.0064	0.37553(80)	$24^3 \times 48$
3.9	0.089	0.0085	0.4302(11)	$24^3 \times 48$



[M. Constantinou et al. (ETMC), *JHEP* 08 (2010) 068 [arXiv:1004.1115]]

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2a^2)$ in Z_q

★ Strategy:

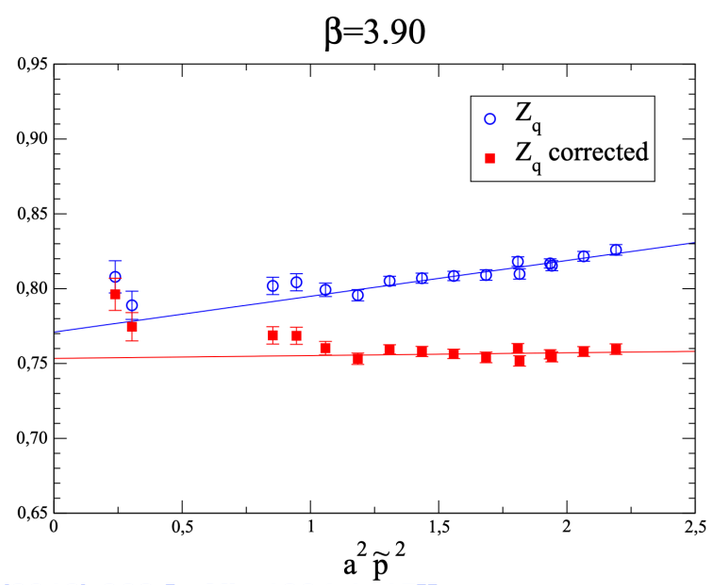
- Calculate numerically the inverse propagator the values of the action parameters and momentum used in non-perturbative estimate

- Calculate Z_q non-perturbatively

- Subtract the artifacts from the estimates

Application for twisted mass fermions and tree-level Symanzik improved gluons

β	a (fm)	$a\mu_0$	m_π (GeV)	$L^3 \times T$
3.9	0.089	0.0040	0.3021(14)	$24^3 \times 48$
3.9	0.089	0.0064	0.37553(80)	$24^3 \times 48$
3.9	0.089	0.0085	0.4302(11)	$24^3 \times 48$



Flatter behavior for corrected estimate of Z_q,
→ better plateau fit

[M. Constantinou et al. (ETMC), *JHEP* 08 (2010) 068 [arXiv:1004.1115]]

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2a^2)$ in Z_q

★ Strategy:

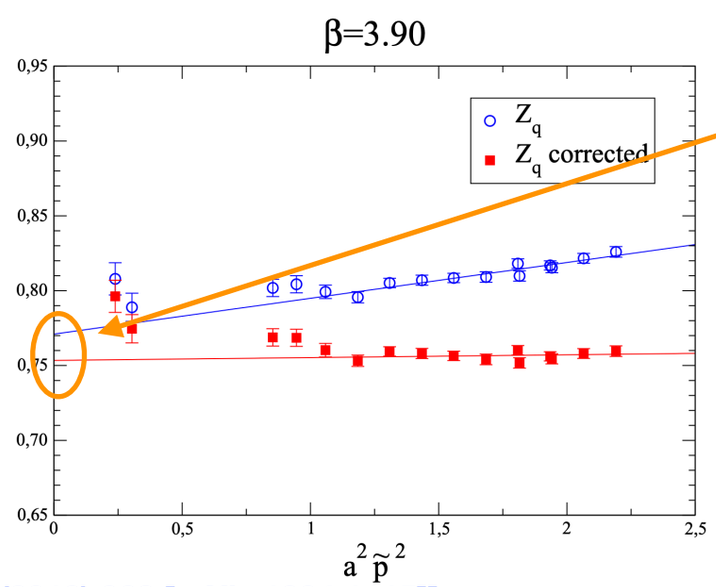
- Calculate numerically the inverse propagator the values of the action parameters and momentum used in non-perturbative estimate

- Calculate Z_q non-perturbatively

- Subtract the artifacts from the estimates

Application for twisted mass fermions and tree-level Symanzik improved gluons

β	a (fm)	$a\mu_0$	m_π (GeV)	$L^3 \times T$
3.9	0.089	0.0040	0.3021(14)	$24^3 \times 48$
3.9	0.089	0.0064	0.37553(80)	$24^3 \times 48$
3.9	0.089	0.0085	0.4302(11)	$24^3 \times 48$



Non-negligible cutoff effects
→ Huge implications in high-precision calculations

Flatter behavior for corrected estimate of Z_q,
→ better plateau fit

[M. Constantinou et al. (ETMC), *JHEP* 08 (2010) 068 [arXiv:1004.1115]]

Contributions of Lattice Pert. Theory (5)

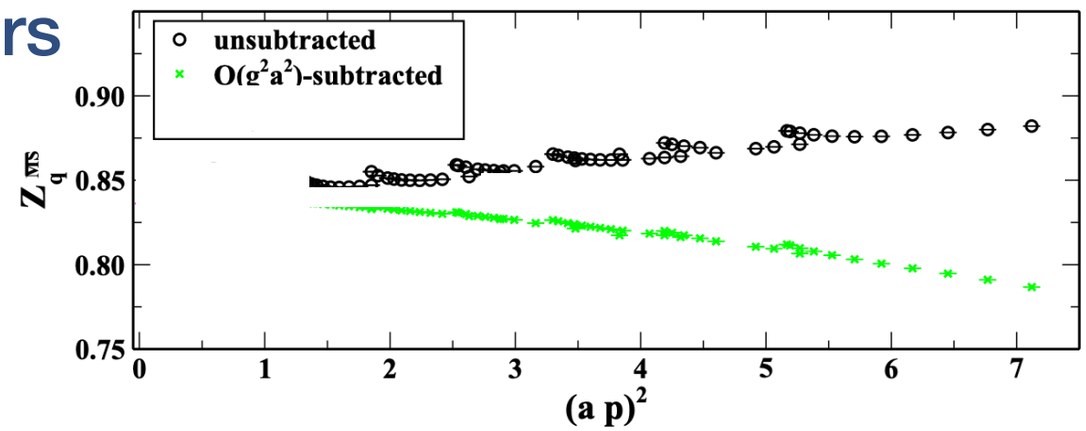
Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

★ $\mathcal{O}(g^2 a^2)$ improvement is not sufficient for all lattice actions and simulation parameters

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

★ $\mathcal{O}(g^2 a^2)$ improvement is not sufficient for all lattice actions and simulation parameters



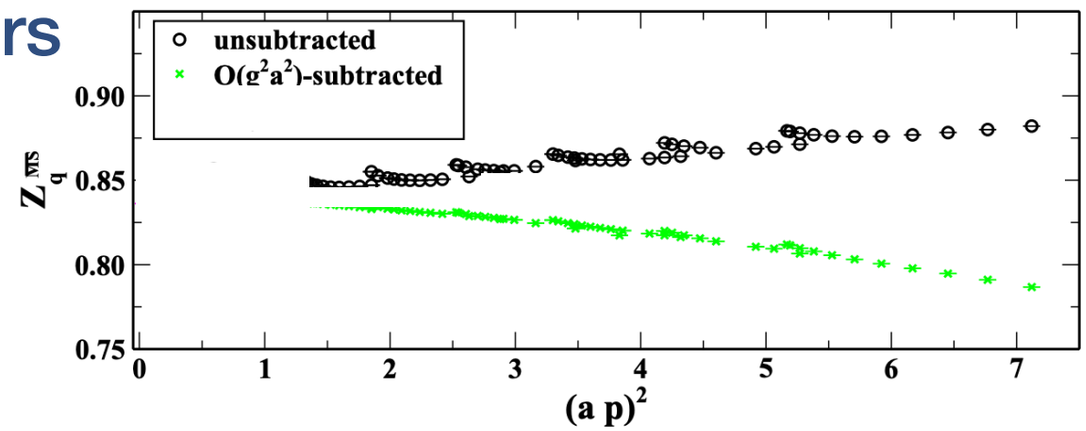
[C. Alexandrou et al. (ETMC), PRD 95 (2017) 3, 034505, [arXiv:1509.00213]]

FIG. 17: Renormalization of the fermion field for N_f=2 twisted mass clover-improved fermions. The data correspond to the $\overline{\text{MS}}$ scheme at a reference scale of 2 GeV and are plotted against the initial renormalization scale, $(a\,\mu)^2 = (a\,p)^2$.

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

★ $\mathcal{O}(g^2 a^2)$ improvement is not sufficient for all lattice actions and simulation parameters



[C. Alexandrou et al. (ETMC), PRD 95 (2017) 3, 034505, [arXiv:1509.00213]]

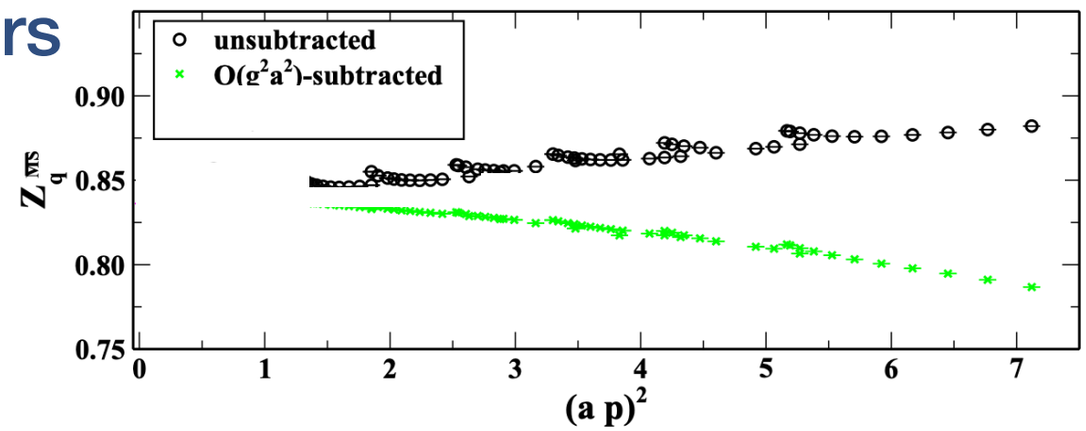
FIG. 17: Renormalization of the fermion field for N_f=2 twisted mass clover-improved fermions. The data correspond to the $\overline{\text{MS}}$ scheme at a reference scale of 2 GeV and are plotted against the initial renormalization scale, $(a\,\mu)^2 = (a\,p)^2$.

★ Alternative approach: Improvement to $\mathcal{O}(g^2 a^\infty)$

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

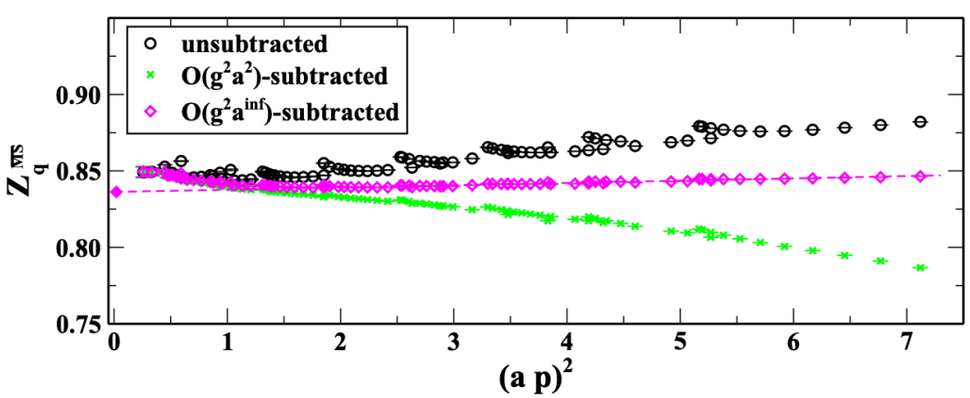
★ $\mathcal{O}(g^2 a^2)$ improvement is not sufficient for all lattice actions and simulation parameters



[C. Alexandrou et al. (ETMC), PRD 95 (2017) 3, 034505, [arXiv:1509.00213]]

FIG. 17: Renormalization of the fermion field for N_f=2 twisted mass clover-improved fermions. The data correspond to the $\overline{\text{MS}}$ scheme at a reference scale of 2 GeV and are plotted against the initial renormalization scale, $(a\,\mu)^2 = (a\,p)^2$.

★ Alternative approach: Improvement to $\mathcal{O}(g^2 a^\infty)$



Subtraction to $\mathcal{O}(g^2 a^\infty)$ leads to a negligible slope because it eliminates cutoff effects

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

★ $\mathcal{O}(g^2 a^2)$ improvement is not sufficient for all lattice actions and simulation parameters

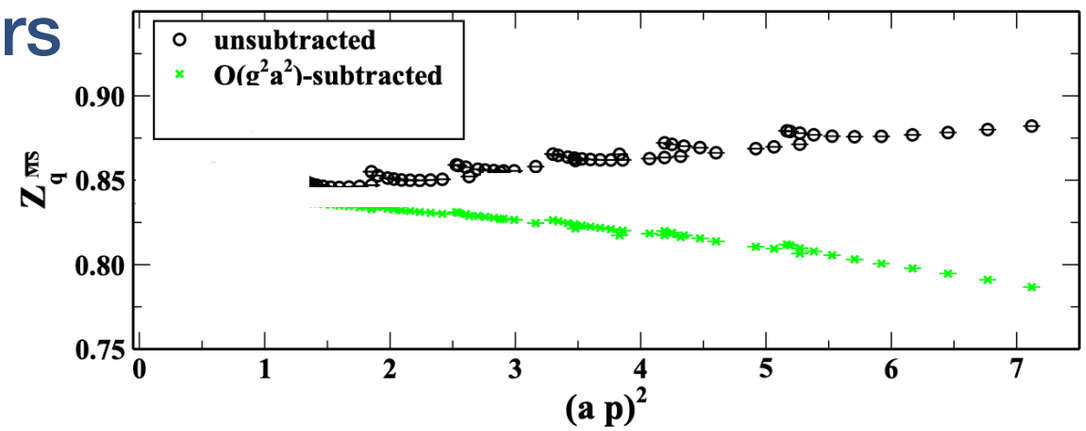
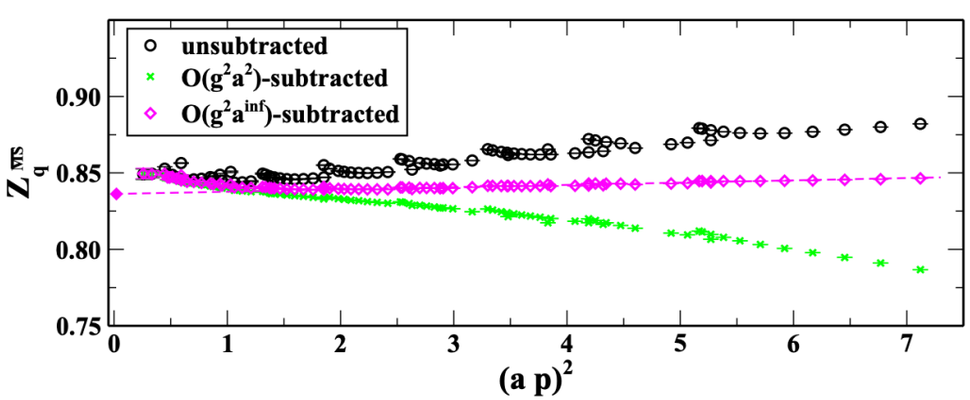


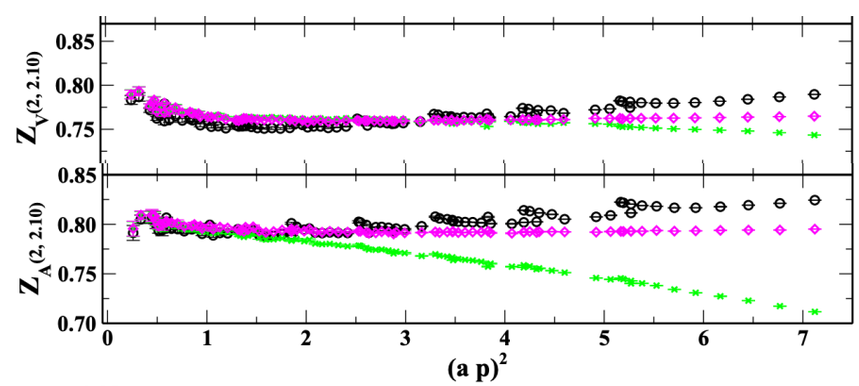
FIG. 17: Renormalization of the fermion field for N_f=2 twisted mass clover-improved fermions. The data correspond to the $\overline{\text{MS}}$ scheme at a reference scale of 2 GeV and are plotted against the initial renormalization scale, $(a\,\mu)^2 = (a\,p)^2$.

[C. Alexandrou et al. (ETMC), PRD 95 (2017) 3, 034505, [arXiv:1509.00213]]

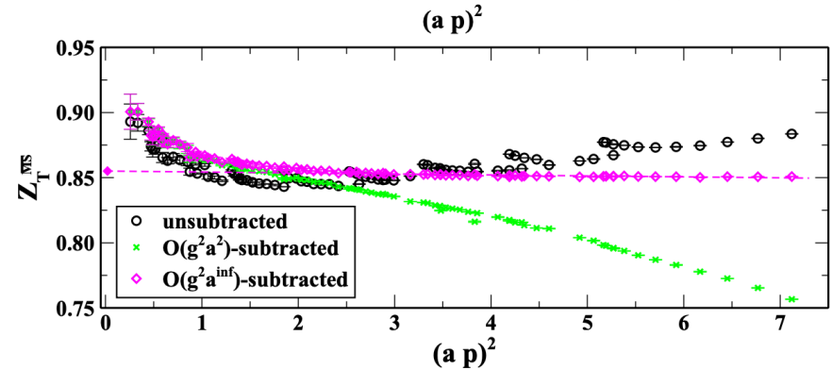
★ Alternative approach: Improvement to $\mathcal{O}(g^2 a^\infty)$



Subtraction to $\mathcal{O}(g^2 a^\infty)$ leads to a negligible slope because it eliminates cutoff effects



Effectiveness of $\mathcal{O}(g^2 a^2)$ depends on the operator under study



$\mathcal{O}(g^2 a^\infty)$-improvement successful for all operators

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

Approach

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

Approach

1. Z_q in RI' scheme for multiple ensembles at different pion mass values. Vertex functions calculated for several momenta (renormalization scales)

$$Z_q = \frac{1}{12}\mathrm{Tr}\left[(S^L)^{-1}(p)\, S^{\mathrm{tree}}(p)\right]\Big|_{p^2=\mu^2}$$

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

Approach

1. Z_q in RI' scheme for multiple ensembles at different pion mass values.
   Vertex functions calculated for several momenta (renormalization scales)

$$Z_q = \frac{1}{12}\mathrm{Tr}\left[(S^L)^{-1}(p)\, S^{\mathrm{tree}}(p)\right]\Big|_{p^2=\mu^2}$$

★ Momenta have to be chosen carefully! $S^{-1}_{(p)} = i\not{p} + \frac{a}{2}p^2 - i\frac{a^2}{6}\not{p}^3 + \cdots - i\,a^2\not{p}\frac{\sum_\mu p_\mu^4}{p^2}\tilde{g}^2\left[-\frac{3}{80} - \frac{C_2}{10} - \frac{5}{48}\lambda\right]$

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

Approach

1. Z_q in RI' scheme for multiple ensembles at different pion mass values. Vertex functions calculated for several momenta (renormalization scales)

$$Z_q = \frac{1}{12}\mathrm{Tr}\left[(S^L)^{-1}(p)\, S^{\mathrm{tree}}(p)\right]\Big|_{p^2=\mu^2}$$

★ Momenta have to be chosen carefully!

$$S^{-1}_{(p)} = i\not{p} + \frac{a}{2}p^2 - i\frac{a^2}{6}\not{p}^3 + \cdots - i\,a^2\not{p}\frac{\sum_\mu p_\mu^4}{p^2}\tilde{g}^2\Big[-\frac{3}{80} - \frac{C_2}{10} - \frac{5}{48}\lambda\Big]$$

$$(ap) \equiv 2\pi\left(\frac{n_t}{L_t} + \frac{1}{2\,L_t}, \frac{n_x}{L_s}, \frac{n_x}{L_s}, \frac{n_x}{L_s}\right) \qquad \sum_i p_i^4/(\sum_i p_i^2)^2 < 0.3$$

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

Approach

1. Z_q in RI' scheme for multiple ensembles at different pion mass values.
   Vertex functions calculated for several momenta (renormalization scales)

$$Z_q = \frac{1}{12}\mathrm{Tr}\left[(S^L)^{-1}(p)\, S^{\mathrm{tree}}(p)\right]\Big|_{p^2=\mu^2}$$

★ Momenta have to be chosen carefully!

$$S^{-1}_{(p)} = i\,\not{p} + \frac{a}{2}p^2 - i\frac{a^2}{6}\not{p}^3 + \cdots - i\,a^2\,\not{p}\,\frac{\sum_\mu p_\mu^4}{p^2}\,\tilde{g}^2\Big[-\frac{3}{80} - \frac{C_2}{10} - \frac{5}{48}\lambda\Big]$$

$$(ap) \equiv 2\pi\left(\frac{n_t}{L_t} + \frac{1}{2\,L_t}, \frac{n_x}{L_s}, \frac{n_x}{L_s}, \frac{n_x}{L_s}\right) \qquad \sum_i p_i^4/(\sum_i p_i^2)^2 < 0.3$$

★ $\mathcal{O}(g^2 a^\infty)$ -subtraction

$$\boxed{Z_q^{\mathrm{RI,\,sub}}(\mu_0, m_\pi) = Z_q^{\mathrm{RI}}(\mu_0, m_\pi, a) - DZ_q(\mu_0, a)}$$

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

Approach

1. Z_q in RI' scheme for multiple ensembles at different pion mass values.
   Vertex functions calculated for several momenta (renormalization scales)

$$Z_q = \frac{1}{12}\mathrm{Tr}\left[(S^L)^{-1}(p)\, S^{\mathrm{tree}}(p)\right]\Big|_{p^2=\mu^2}$$

★ Momenta have to be chosen carefully!

$$S^{-1}_{(p)} = i\not{p} + \frac{a}{2}p^2 - i\frac{a^2}{6}\not{p}^3 + \cdots - i\,a^2\not{p}\frac{\sum_\mu p_\mu^4}{p^2}\tilde{g}^2\Big[-\frac{3}{80} - \frac{C_2}{10} - \frac{5}{48}\lambda\Big]$$

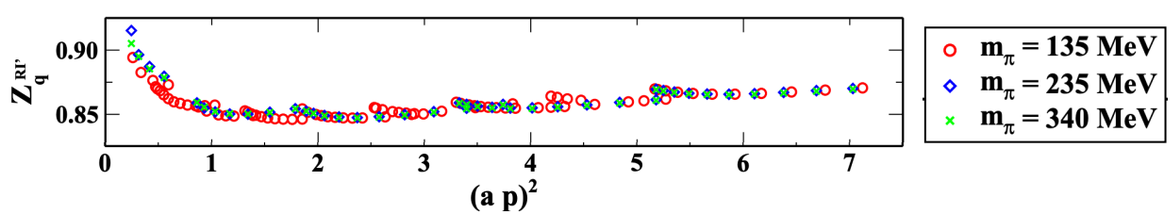
$$(ap) \equiv 2\pi\left(\frac{n_t}{L_t} + \frac{1}{2\,L_t}, \frac{n_x}{L_s}, \frac{n_x}{L_s}, \frac{n_x}{L_s}\right) \qquad \sum_i p_i^4/(\sum_i p_i^2)^2 < 0.3$$

★ $\mathcal{O}(g^2 a^\infty)$ -subtraction

$$Z_q^{\mathrm{RI,\,sub}}(\mu_0, m_\pi) = Z_q^{\mathrm{RI}}(\mu_0, m_\pi, a) - DZ_q(\mu_0, a)$$

2. Chiral extrapolation

$$Z_q^{\mathrm{RI}}(\mu_0, m_\pi) = \mathcal{Z}_q^{\mathrm{RI}}(\mu_0) + m_\pi^2\,\bar{Z}_q^{\mathrm{RI}}(\mu_0)$$



Nf=2 twisted mass fermions

$a\mu$	κ	aM_{PS}	lattice size
$\beta = 2.10,\ a = 0.093$ fm, $c_{\mathrm{sw}} = 1.57551$			
0.0009	0.13729	0.0621(2)	$48^3 \times 96$
0.0030	0.1373	0.110(4)	$24^3 \times 48$
0.0060	0.1373	0.160(4)	$24^3 \times 48$

Contributions of Lattice Pert. Theory (5)

Improvement to $\mathcal{O}(g^2 a^\infty)$ in Z_q

Approach

1. Z_q in RI' scheme for multiple ensembles at different pion mass values. Vertex functions calculated for several momenta (renormalization scales)

$$Z_q = \frac{1}{12}\mathrm{Tr}\left[(S^L)^{-1}(p)\, S^{\mathrm{tree}}(p)\right]\Big|_{p^2=\mu^2}$$

★ Momenta have to be chosen carefully!

$$S^{-1}_{(p)} = i\not{p} + \frac{a}{2}p^2 - i\frac{a^2}{6}\not{p}^3 + \cdots - i\,a^2\not{p}\frac{\sum_\mu p_\mu^4}{p^2}\tilde{g}^2\Big[-\frac{3}{80} - \frac{C_2}{10} - \frac{5}{48}\lambda\Big]$$

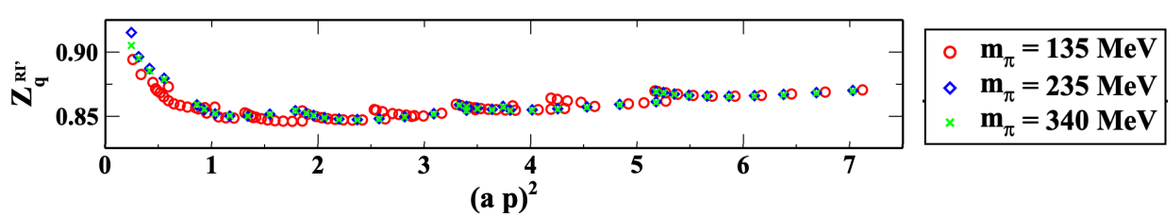
$$(ap) \equiv 2\pi\left(\frac{n_t}{L_t} + \frac{1}{2\,L_t}, \frac{n_x}{L_s}, \frac{n_x}{L_s}, \frac{n_x}{L_s}\right) \qquad \sum_i p_i^4/(\sum_i p_i^2)^2 < 0.3$$

★ $\mathcal{O}(g^2 a^\infty)$ -subtraction

$$Z_q^{\mathrm{RI,\,sub}}(\mu_0, m_\pi) = Z_q^{\mathrm{RI}}(\mu_0, m_\pi, a) - DZ_q(\mu_0, a)$$

2. Chiral extrapolation

$$Z_q^{\mathrm{RI}}(\mu_0, m_\pi) = \mathcal{Z}_q^{\mathrm{RI}}(\mu_0) + m_\pi^2\,\bar{Z}_q^{\mathrm{RI}}(\mu_0)$$

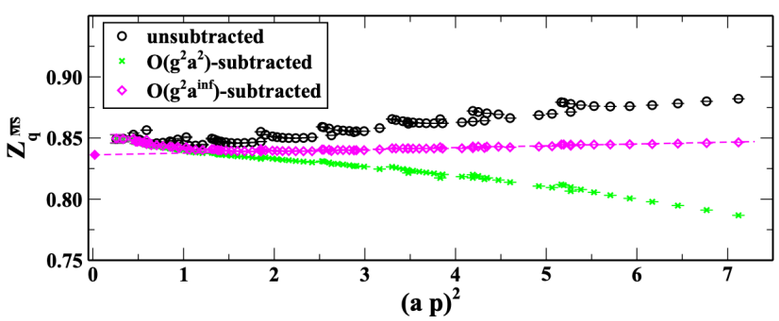


Nf=2 twisted mass fermions

$a\mu$	κ	aM_{PS}	lattice size
$\beta = 2.10,\ a = 0.093$ fm, $c_{\mathrm{sw}} = 1.57551$			
0.0009	0.13729	0.0621(2)	$48^3 \times 96$
0.0030	0.1373	0.110(4)	$24^3 \times 48$
0.0060	0.1373	0.160(4)	$24^3 \times 48$

3. Conversion to $\overline{\mathrm{MS}}$

$$Z_q^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = C^{\overline{\mathrm{MS}},\,\mathrm{RI}}(2\,\mathrm{GeV}, \mu_0)\,\mathcal{Z}_q^{\mathrm{RI}}(\mu_0)$$

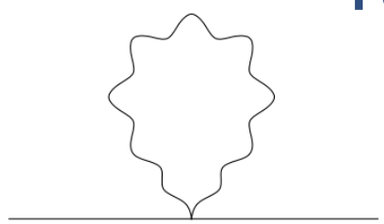


[C. Alexandrou et al. (ETMC), PRD 95 (2017) 3, 034505, [arXiv:1509.00213]]

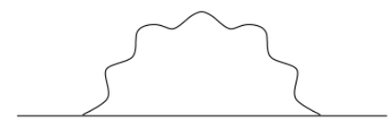
Renormalization of fermion field

1-loop lattice perturbation theory with Wilson-type fermions

Feynman diagrams



1



2

★ Two types of vertices

$$\bar{\psi}\psi A = g\,\delta(k_1 - k_2 + k_3)\left(i\cos(\frac{(k_2+k_3)_\mu}{2})\gamma^\mu + \sin(\frac{(k_2+k_3)_\mu}{2})\hat{1} - c_{\rm SW}\cos(\frac{(k_1)_\mu}{2})[\gamma^\nu, \sin(\frac{(k_1)_\rho}{2})\,\gamma^\rho]/4 \right)$$

$$\bar{\psi}\psi AA = g^2\,\delta(k_1 + k_2 - k_3 + k_4)\left(i\frac{1}{2}\delta_{\mu\nu}\sin(\frac{(k_3+k_4)_\mu}{2})\gamma^\mu + \frac{1}{2}\delta_{\mu\nu}\cos(\frac{(k_3+k_4)_\mu}{2})\hat{1} + 12 \text{ terms with } c_{\rm SW} \right)$$

$$d1 = \frac{g^2}{8}\frac{N_c^2-1}{N_c}\int dk\left[\frac{2i\gamma^\rho\sin(q_\rho) + 2\hat{1}cos(q_\rho)}{prop(k)_{\rho\rho}} + c_{\rm SW}\frac{\cos(\frac{k_\rho}{2})\cos(\frac{k_\sigma}{2})[\gamma^\rho,\gamma^\sigma]}{prop(k)_{\rho\sigma}}\right] \;:\; \text{convergent}$$

$$d2 = g^2\frac{N_c^2-1}{N_c}\left[aLog[q^2] + b\right]$$

D2 contains divergences

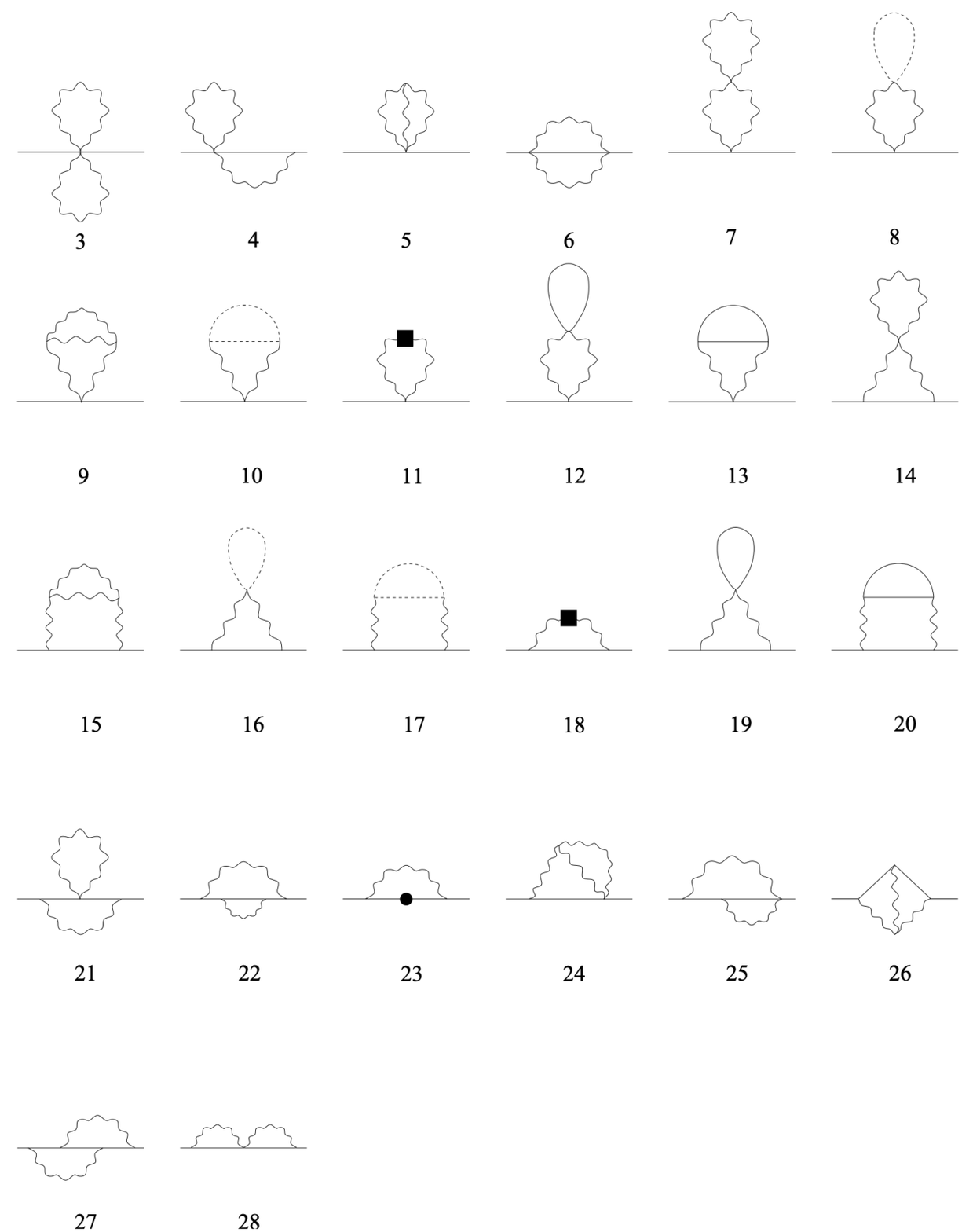
Renormalization of fermion field

Extension to 2-loops

- ★ **Several complications compared to 1-loop:**
  - **More Feynman diagrams**
  - **Vertices with more gluons**
  - **parameters like coupling constant, gauge parameter, must be renormalized beyond tree-level (unlike 1-loop level)**

$$(1 + g^2 a_1 + g^4 a_2 + \cdots)(1 + g^2 b_1 + g^4 b_2 + \cdots) =$$

$$1 + g^2(a_1 + b1) + g^4(a_2 + b_2 + 2a_1 b_1) + \mathcal{O}(g^6)$$



Wavy (solid, dotted) lines represent gluons (fermions, ghosts). Solid boxes denote vertices stemming from the measure part of the action; a solid circle is a fermion mass counterterm.

Thank you