Excercises for "QCD (and QED) in a Euclidean Finite-Volume" Day 1, EuroPLEx 2021

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1 Euclidean vs. Minkowski two-point function

1.1 Minkowski

Begin with the definition

$$\widetilde{G}_{\mathsf{M}}(p_{\mathsf{M}}) = \int d^4x_{\mathsf{M}}\, e^{-ip_{\mathsf{M}}\cdot x_{\mathsf{M}}}\, \langle 0|\mathrm{T}\{\phi(x_{\mathsf{M}})\phi(0)\}|0\rangle\,, \tag{1}$$

where $p_{\mathsf{M}}\cdot x_{\mathsf{M}} = -p^0x^0 + \boldsymbol{p}\cdot\boldsymbol{x}$ and

$$\langle 0|\mathrm{T}\{\phi(x_{\mathsf{M}})\phi(0)\}|0\rangle = \langle 0|\phi(0,\boldsymbol{x})e^{-i\hat{H}x^0-\epsilon x^0}\phi(0)|0\rangle\,, \quad \text{for } x^0>0\,, \tag{2}$$

$$\langle 0|\mathrm{T}\{\phi(x_{\mathsf{M}})\phi(0)\}|0\rangle = \langle 0|\phi(0)e^{+i\hat{H}x^0+\epsilon x^0}\phi(0,\boldsymbol{x})|0\rangle\,, \quad \text{for } x^0<0\,. \tag{3}$$

Insert the identity, defined as

$$\mathbb{I} = \sum_N \frac{1}{N!}\prod_{i=1}^N \int \frac{d^3\boldsymbol{p}_i}{(2\pi)^3 2\omega_{\boldsymbol{p}_i}} |\boldsymbol{p}_1,\cdots,\boldsymbol{p}_n\rangle\langle \boldsymbol{p}_1,\cdots,\boldsymbol{p}_n|\,, \tag{4}$$

$$= \int d\alpha \int \frac{d^3\boldsymbol{P}}{(2\pi)^3 2\omega_{\boldsymbol{P},\alpha}} |\alpha,\boldsymbol{P}\rangle\langle\alpha,\boldsymbol{P}|\,, \tag{5}$$

where α is a collective index and

$$\hat{H}|\alpha,\boldsymbol{P}\rangle = |\alpha,\boldsymbol{P}\rangle\sqrt{M_\alpha^2+\boldsymbol{P}^2} = |\alpha,\boldsymbol{P}\rangle\omega_{\boldsymbol{P},\alpha}\,, \tag{6}$$

$$\hat{\boldsymbol{P}}|\alpha,\boldsymbol{P}\rangle = |\alpha,\boldsymbol{P}\rangle\boldsymbol{P}\,. \tag{7}$$

Demonstrate

$$\widetilde{G}_{\mathsf{M}}(p_{\mathsf{M}}) = \frac{1}{i}\int d\alpha \frac{|Z(\alpha)|^2}{p_{\mathsf{M}}^2+M_\alpha^2-i\epsilon}\,, \tag{8}$$

and give the expression for $Z(\alpha)$.

1.2 Euclidean

Now take

$$\widetilde{G}_{\mathsf{E}}(p_{\mathsf{E}}) = \int d^4x_{\mathsf{E}}\, e^{-ip_{\mathsf{E}}\cdot x_{\mathsf{E}}}\, \langle 0|\mathrm{T}\{\phi(x_{\mathsf{E}})\phi(0)\}|0\rangle\,, \tag{9}$$

where $p_{\mathsf{E}} \cdot x_{\mathsf{E}} = p_4 x_4 + \boldsymbol{p}\cdot\boldsymbol{x}$ and

$$\langle 0|\mathrm{T}\{\phi(x_{\mathsf{E}})\phi(0)\}|0\rangle = \langle 0|\phi(0,\boldsymbol{x})e^{-\hat{H}x_4}\phi(0)|0\rangle\,, \quad \text{for } x_4 > 0\,, \tag{10}$$

$$\langle 0|\mathrm{T}\{\phi(x_{\mathsf{M}})\phi(0)\}|0\rangle = \langle 0|\phi(0)e^{+\hat{H}x_4}\phi(0,\boldsymbol{x})|0\rangle\,, \quad \text{for } x_4 < 0\,, \tag{11}$$

and derive the analogous result.

2 Twisted, asymmetric boundary conditions

Suppose we take the boundary conditions

$$\pi(\tau,\boldsymbol{x})e^{i\theta_i} = \pi(\tau,\boldsymbol{x}+L_i\boldsymbol{e}_i)\,, \tag{12}$$

where $\boldsymbol{e}_i$ is a unit vector in the ith direction, $i \in \{x,y,z\}$. Derive the discrete set of momenta, $\boldsymbol{p}$, such that the Fourier transform respects the boundary conditions.

3 Leading-order finite-volume mass in $\lambda\phi^4$ theory

3.1 Evaluating the Bessel integral

In lecture we found the mass shift for $\lambda\phi^4$ theory in a periodic cubic volume:

$$\Delta m^2(L) \equiv m(L)^2 - m^2 = \frac{\lambda_0}{2}\Delta_{\boldsymbol{k}} \int \frac{dk_4}{2\pi}\frac{1}{k^2+m^2}\,, \tag{13}$$

where $k^2 = k_4^2 + \boldsymbol{k}^2$ and

$$\Delta_{\boldsymbol{k}} \equiv \left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right]. \tag{14}$$

Apply the Poisson summation formula and cast the integral into the form

$$K_1(z) = \frac{z}{4}\int_0^\infty \frac{dt}{t^2}\exp[-t - z^2/(4t)]\,, \tag{15}$$

to evaluate.

3.2 Zero-mode quenching

Suppose we modify the theory such that

$$\Delta_{\boldsymbol{k}} \longrightarrow \Delta'_{\boldsymbol{k}} \equiv \left[\frac{1}{L^3}\sum_{\boldsymbol{k}\neq\boldsymbol{0}} - \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right]. \tag{16}$$

Give the full result for the $\mathcal{O}(\lambda)$ value of $\Delta m^2(L)$ in this case.

4 Single-particle matrix element in $g\phi^3$

4.1 Finite-volume decay constant

Define $j(x) = Z_j\phi(x)$ and define

$$f_\phi(L)^2 = |\langle 0|j(0)|\phi, L, \boldsymbol{p}=\boldsymbol{0}\rangle|^2\,, \tag{17}$$

$$f_\phi(\infty)^2 = |\langle 0|j(0)|\phi\rangle|^2\,. \tag{18}$$

Determine how to extract $f_\phi(L)^2$ from

$$\widetilde{G}^f_L(p^0) = Z_j^2 \int d^3\boldsymbol{x} \int dx^0\, e^{ip^0x^0}\, \langle 0|\mathrm{T}\{\phi(x^0,\boldsymbol{x})\phi(0)\}|0\rangle_L\,, \tag{19}$$

and give the analogous expression for $f_\phi(\infty)$.

4.2 Self energy summation

Show that the sum of all diagrams, to all orders can be written as

$$\widetilde{G}^f_L(p^0) = \frac{1}{i}\frac{Z_j^2}{-(p^0)^2 + m_0^2 + \Sigma_L(-(p^0)^2)}\,, \tag{20}$$

and define Σ_L.

4.3 Volume effects

Show that

$$\frac{f_\phi(L)^2}{f_\phi(\infty)^2} = 1 - \left[\Sigma'_L(-m^2) - \Sigma'_\infty(-m^2)\right] + \mathcal{O}(g^3)\,, \tag{21}$$

and evaluate the g^2 result.