UK Research and Innovation

QCD (and QED) in a Euclidean finite volume

EuroPLEx Summer School 2021

EuroPLEx

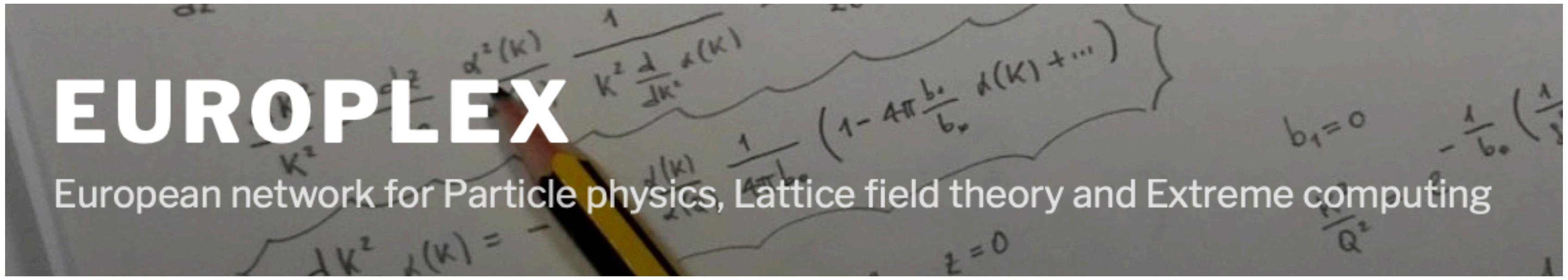
H2020

Maxwell T. Hansen

August 23rd-27th

THE UNIVERSITY *of* EDINBURGH

Welcome to the EuroPLEx Summer School…

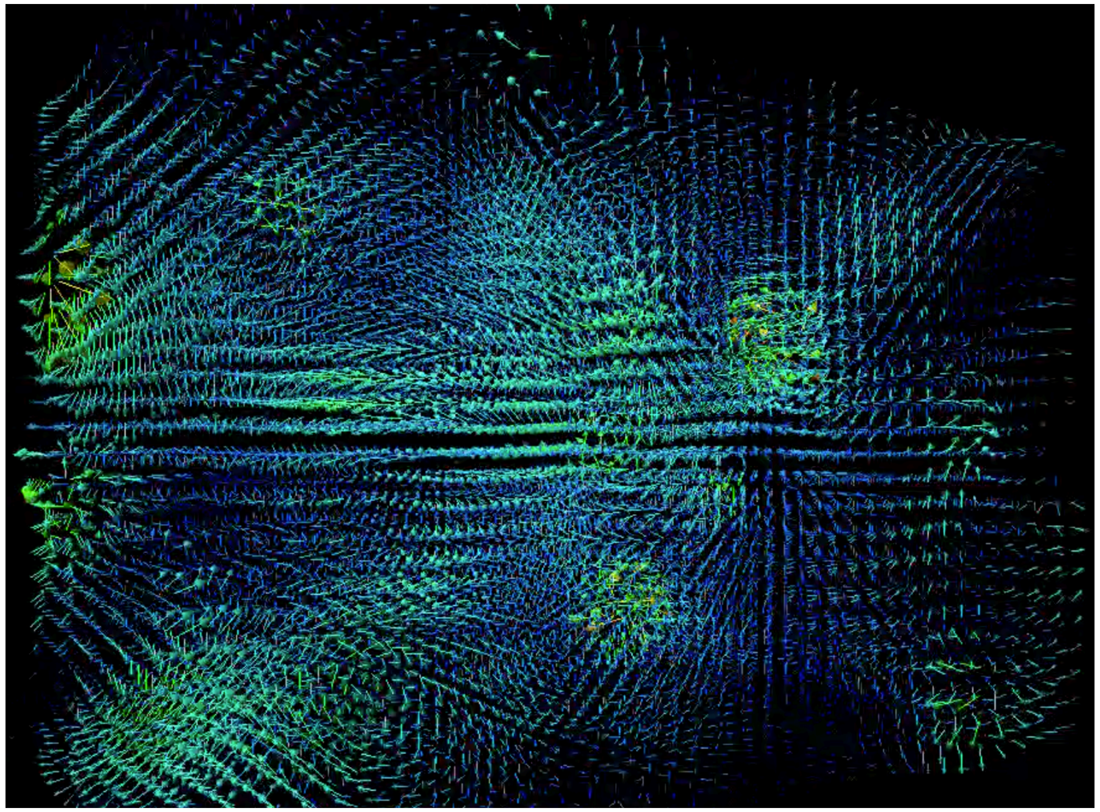


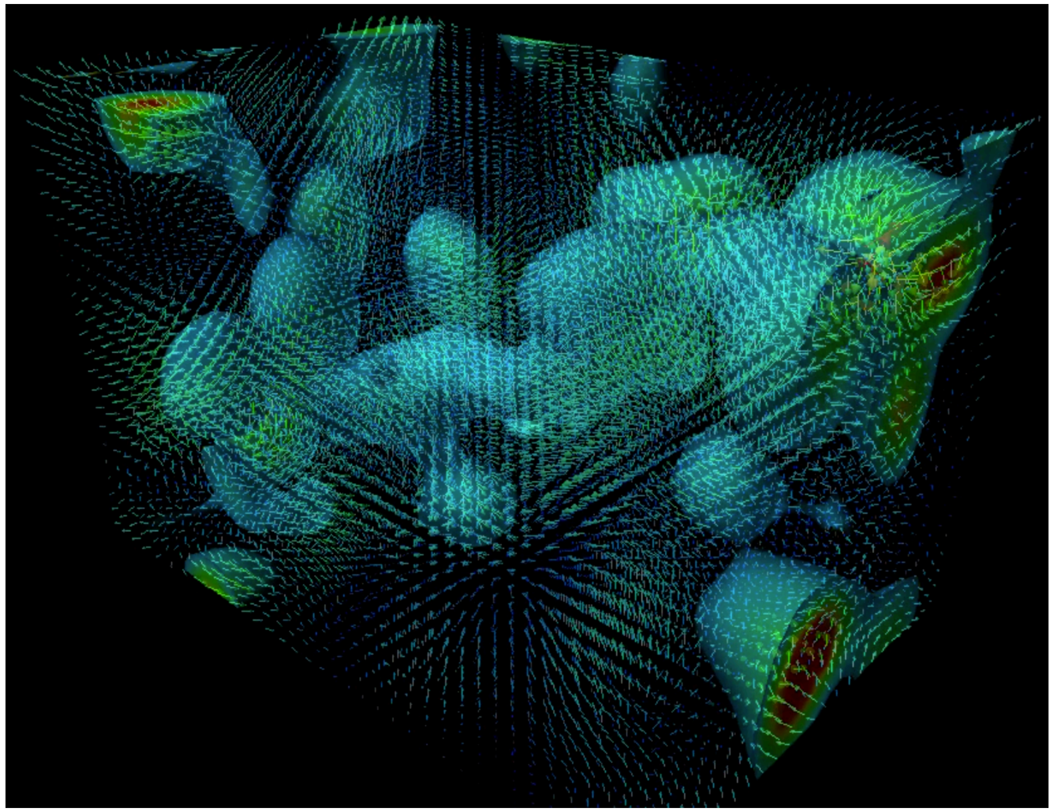
EuroPLEx will provide a stimulating and fertile environment to train a new generation of researchers in Theoretical Particle Physics, equipped with all the analytical and computational skills that are distinctive of the field of Lattice QCD.

… and welcome to virtual Edinburgh!

Lattice field theory

- Non-perturbative regulator of quantum field theory (QFT)
- Systematically improvable numerical method for extracting QFT's properties
- Exciting, vibrant, highly active research community
- Technical field that challenges all of us to be great communicators



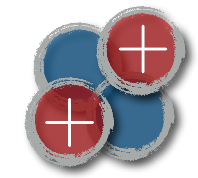


University of Adelaide, CSSM

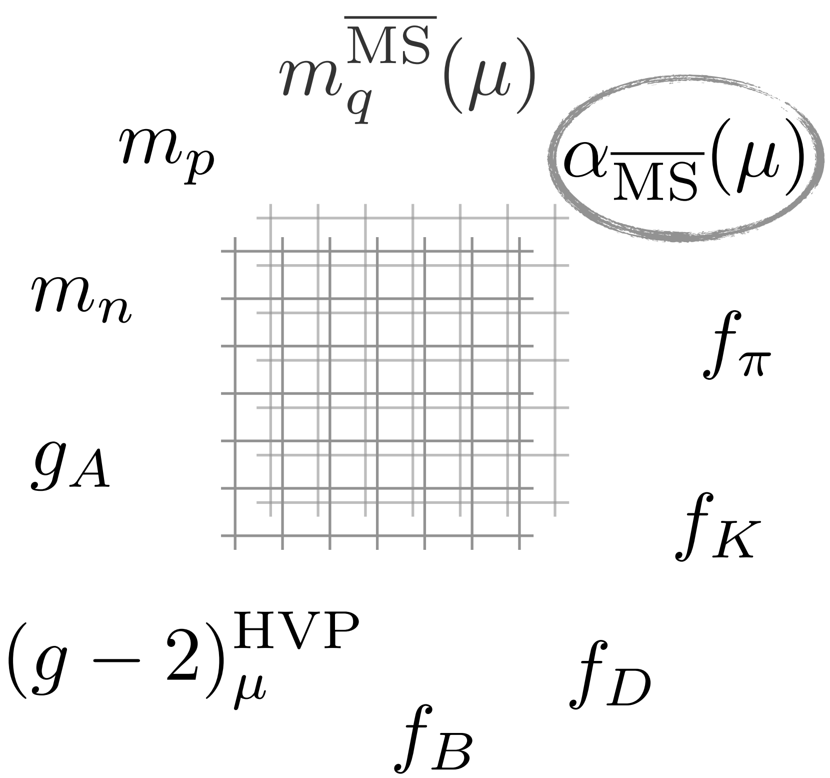
Lattice QCD

1. Lagrangian defining QCD +
2. Formal / numerical machinery (lattice QCD) +
3. A few experimental inputs (e.g. M_π, M_K, M_Ω) =

$$\mathcal{L}_{\rm QCD} = \sum_f \overline{\Psi}_f (i\not{D} - m_f)\Psi_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$



Wide range of precision pre-/post-dictions



FLAG 2019
Flavour Lattice Averaging Group

$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1182(8)$ lattice average

$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1174(16)$ PDG 18 (non-lattice)

Overwhelming evidence for QCD ✓ → *Tool for new physics searches*

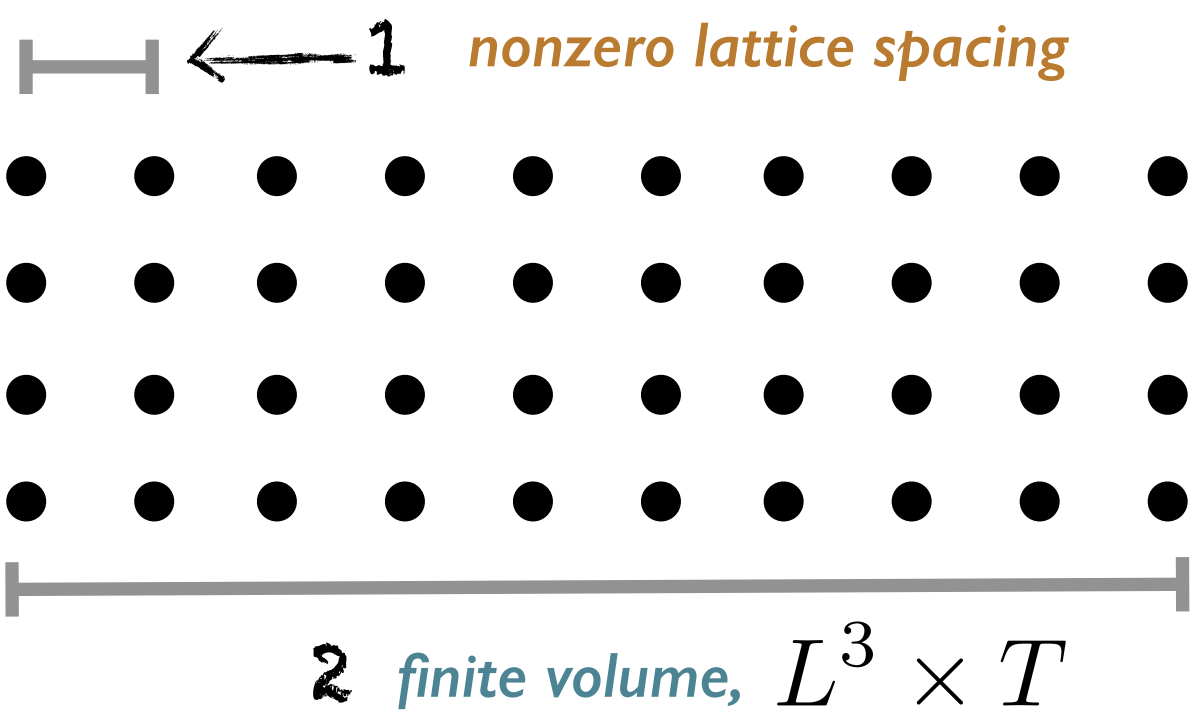
Three essential modifications

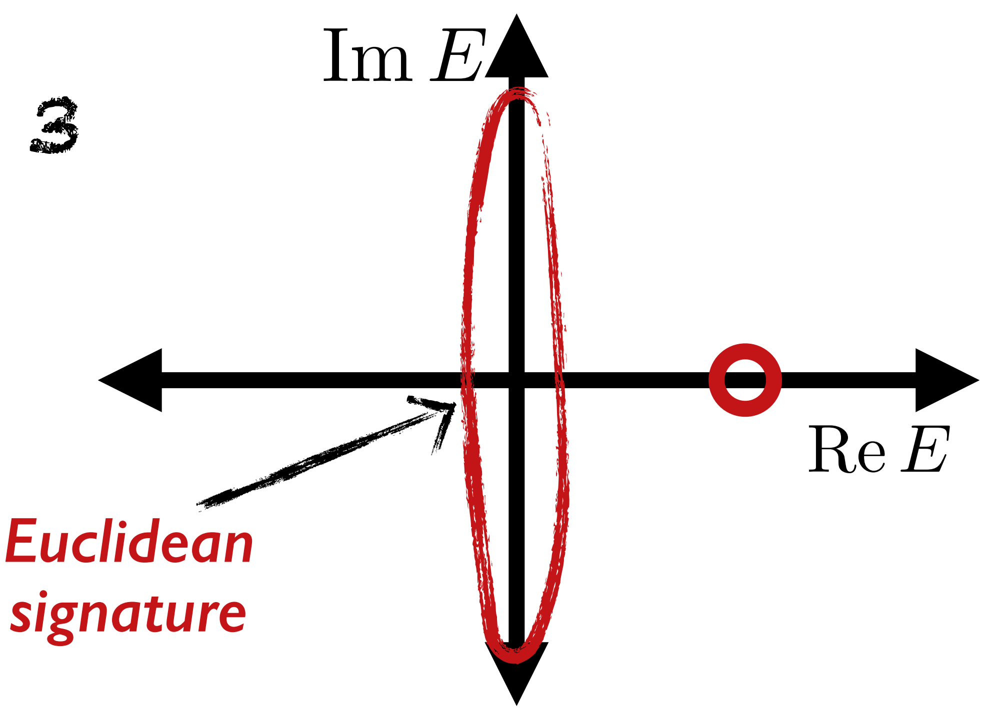
$$\text{observable} = \int \mathcal{D}\phi \; e^{iS} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

Three essential modifications

$$\text{observable?} = \int d^N\!\phi \, e^{-S} \left[\begin{array}{c}\textbf{interpolator} \\ \textbf{for observable}\end{array} \right]$$

To proceed we have to make *three modifications*

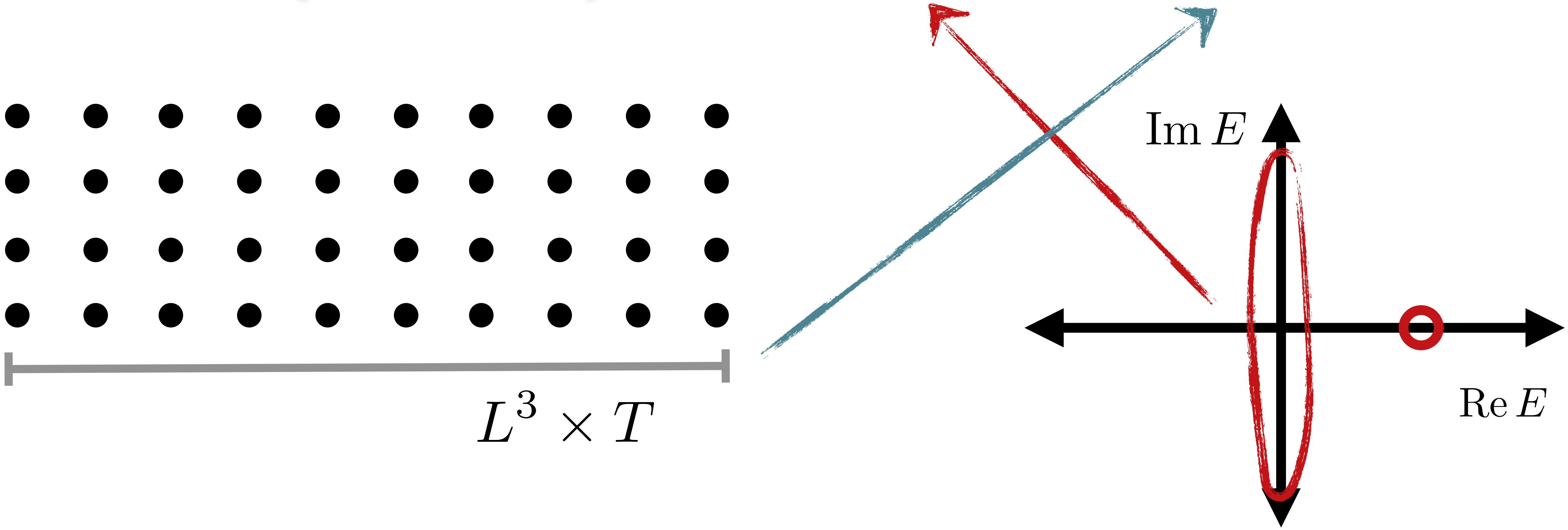




Also… $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

(but physical masses → increasingly common)

QCD (and QED) in a Euclidean finite volume



Sometimes the **finite-volume** is an **unwanted artifact** (extrapolate $L \to \infty$)…

… but it can also be a **useful tool** (fit data to predicted L dependence)

Using the volume as a tool often resolves the issue of Euclidean signature

Five-day outline

- ☐ Warm-up and definitions
  - ☐ Meaning of Euclidean
  - ☐ Finite-volume set-up
- ☐ e^{-mL} volume effects
  - ☐ Mass, matrix element in $\lambda\phi^4, g\phi^3$
  - ☐ Pion mass
  - ☐ LO-HVP for $(g-2)_\mu$
  - ☐ Bethe-Salpeter kernel
- ☐ $2 \to 2$ finite-volume (f.v.) formalism
  - ☐ Scattering basics
  - ☐ Derivation
  - ☐ Generalizations
  - ☐ Applications
- ☐ $(1+)\mathscr{J} \to 2$ f.v. formalism
  - ☐ Derivation
  - ☐ Expansions
  - ☐ Applications
- ☐ $2 + \mathscr{J} \to 2$ f.v. formalism
  - ☐ Derivation
  - ☐ Testing the result
  - ☐ Numerical explorations
- ☐ Non-local matrix elements
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  - ☐ Applications
- ☐ $3 \to 3$ f.v. formalism
  - ☐ New complications
  - ☐ Derivation ($E_n(L)$ to $\mathscr{K}_{\text{df},3}$)
  - ☐ Integral equations ($\mathscr{K}_{\text{df},3}$ to $\mathscr{M}_3$)
  - ☐ Testing the result
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- ☐ Finite-volume QED + QCD
  - ☐ Different formalisms
  - ☐ Basic examples

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Euclidean vs Minkowski

Four-vector basics

$$x_M^\alpha = (t,\vec{x})^\alpha, \quad p_M^\alpha = (E,\vec{p})^\alpha$$

$$x_M\cdot p_M = x_M^\alpha p_{M,\alpha} = -Et + \vec{p}\cdot\vec{x}$$

$$x_{E,\alpha} = (\vec{x}, it) = (\vec{x},\tau)_\alpha \qquad \alpha = 1,2,3,4$$

$$x_E^2 = \tau^2 + \vec{x}^2$$

Minkowski two-point function (generic scalar theory)

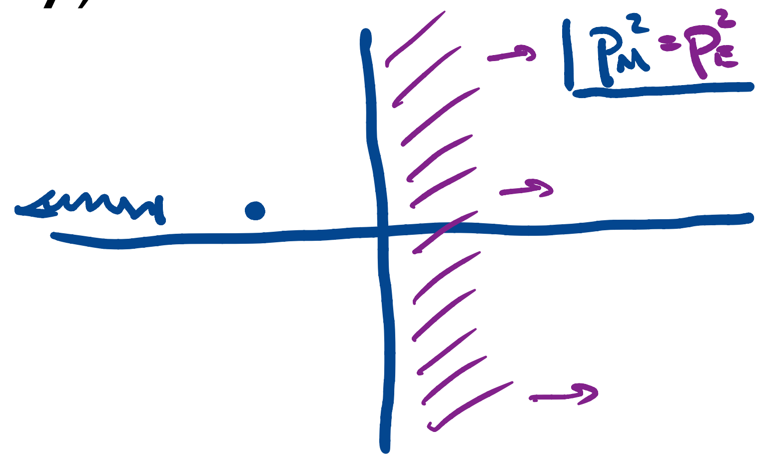
$$\tilde{G}_M(p_M) = \int d^4x_M\, e^{-ip_M\cdot x_M}\langle 0|T\{\phi(x_M)\phi(0)\}|0\rangle$$

$$= \int_0^\infty dt\, e^{+iEt}\langle 0|\tilde{\phi}(0,\vec{p})\, e^{-i\hat{H}t-\epsilon t}\,\phi(0)|0\rangle + \cdots$$

$$= \int d\alpha\, \frac{1}{2\omega_{\alpha,\vec{p}}}\,\frac{1}{i}\,\frac{|Z_\alpha|^2}{-E+\omega_{\alpha,\vec{p}}-i\epsilon} + \cdots$$

$$Z_\alpha = \langle 0|\phi(0)|\alpha,\vec{p}\rangle, \quad \omega_{\alpha,\vec{p}} = \sqrt{\vec{p}^2 + M_\alpha^2}$$

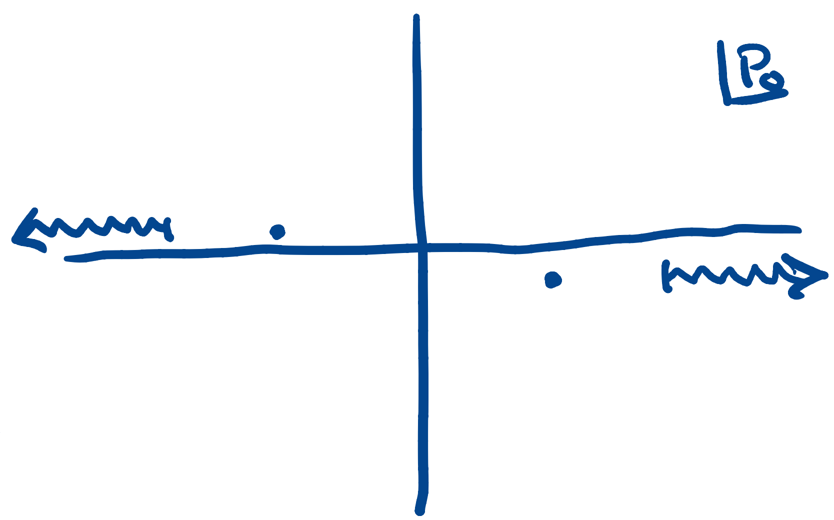
$$= 1/i \int d\alpha\, \frac{|Z_\alpha|^2}{p_M^2 + M_\alpha^2 - i\epsilon}$$

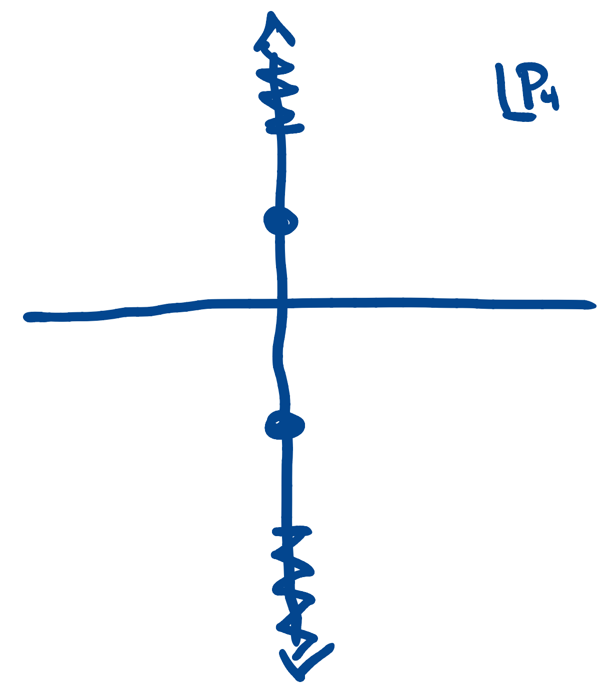


Euclidean two-point function

$$\tilde{G}_E(p_E) = \int d^4x_E\, e^{-ip_E\cdot x_E}\langle \ldots \rangle = \int d\alpha\, \frac{|Z_\alpha|^2}{p_E^2 + M_\alpha^2}$$

$$e^{-\hat{H}\tau}$$





Meaning of Euclidean

When asked **"Is it Euclidean or Minkowski?"** first think:

"Is the question well-posed?"

Sometimes it is

- ❑ Correlation functions
- ❑ Four-vectors
- ❑ QFT path integral

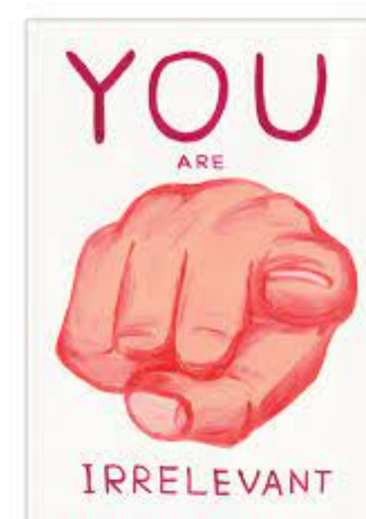
Often it is not

- ❑ Masses
- ❑ Decay constants
- ❑ Finite-volume energies
- ❑ Scattering amplitudes
- ❑ Local matrix elements
- ❑ Spectral densities

For the case of **correlation functions**, Minkowski vs Euclidean is…

Important **when we have limited knowledge**
(e.g. numerical estimate for real Euclidean momenta)

Irrelevant when we have **full analytic knowledge**

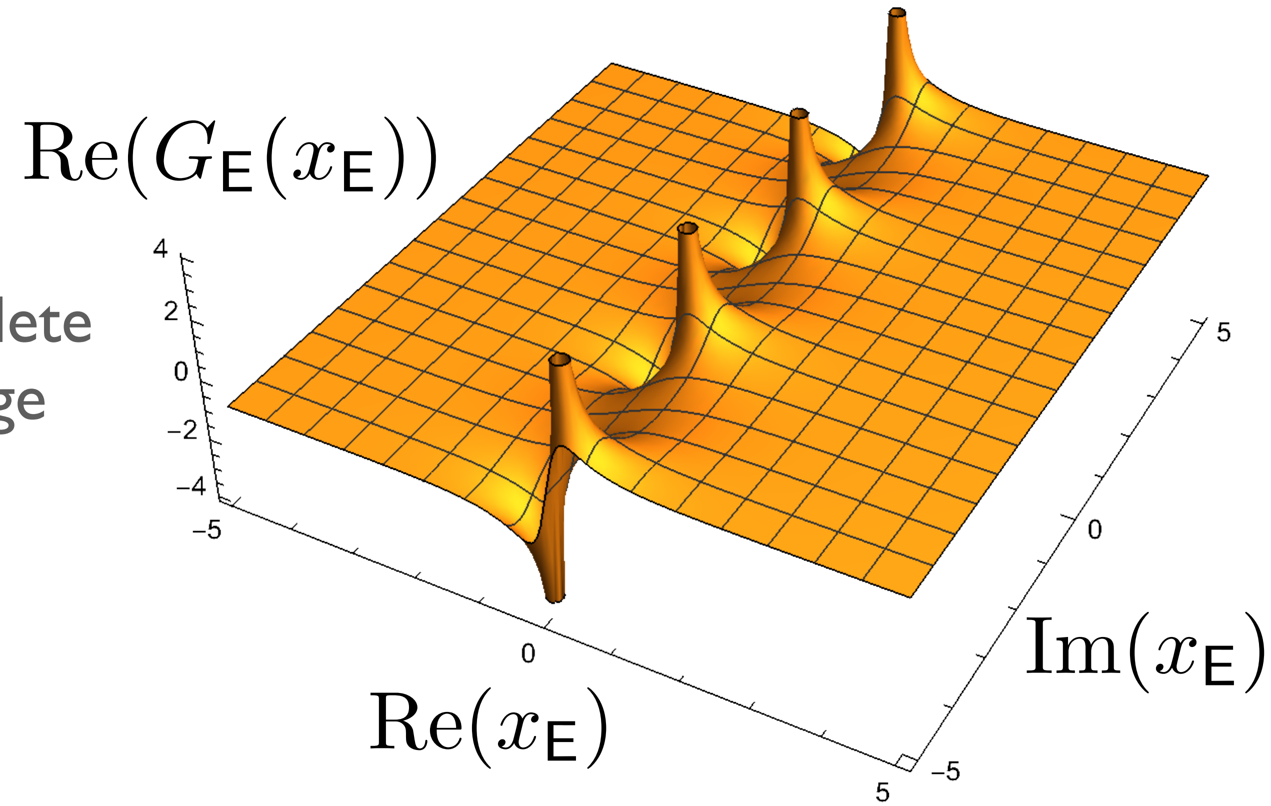


Analytic knowledge

Suppose the Euclidean correlator is

$$G_{\mathsf{E}}(x_{\mathsf{E}}) = \tanh(x_{\mathsf{E}})$$

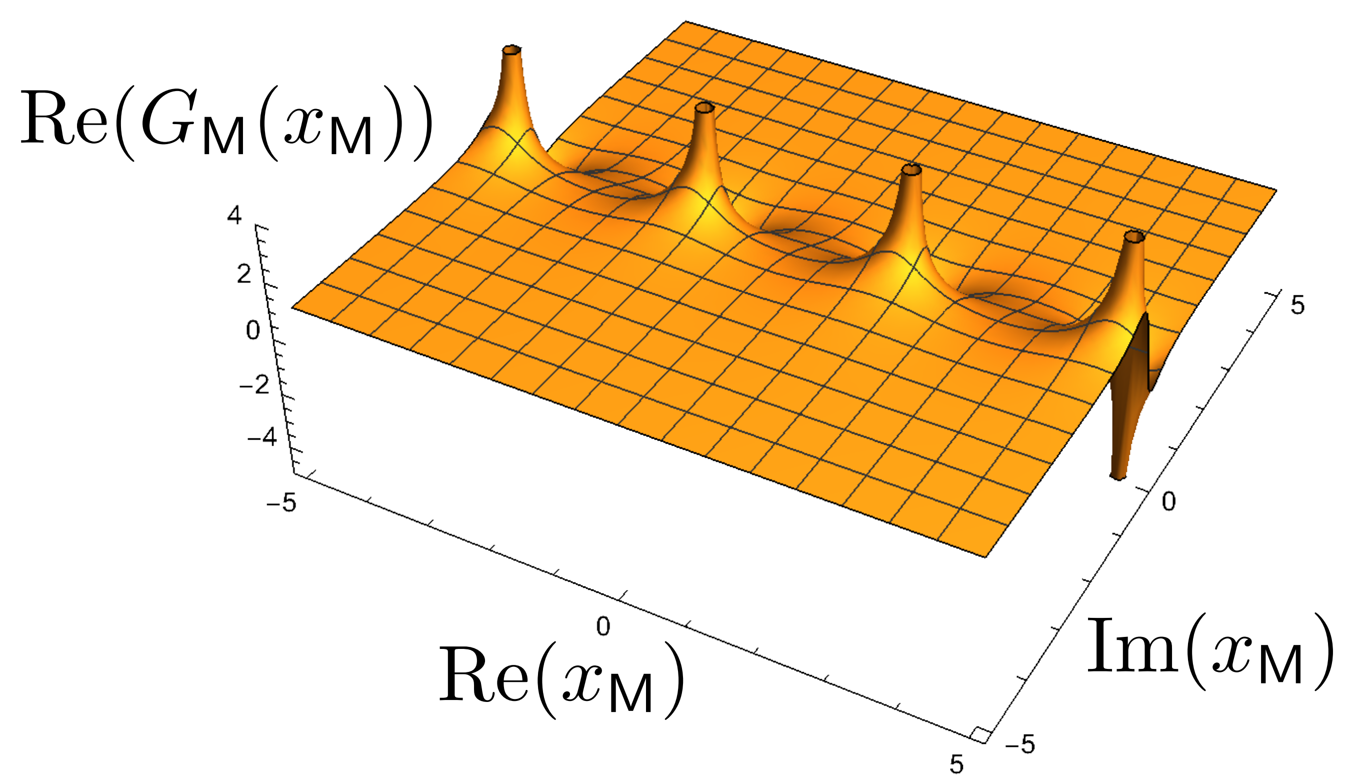
and we have complete analytic knowledge



… analytic continuation gives the Minkowski correlator

$$G_{\mathsf{M}}(x_{\mathsf{M}}) = i\tan(x_{\mathsf{M}})$$

but this is just a relabeling of the same information

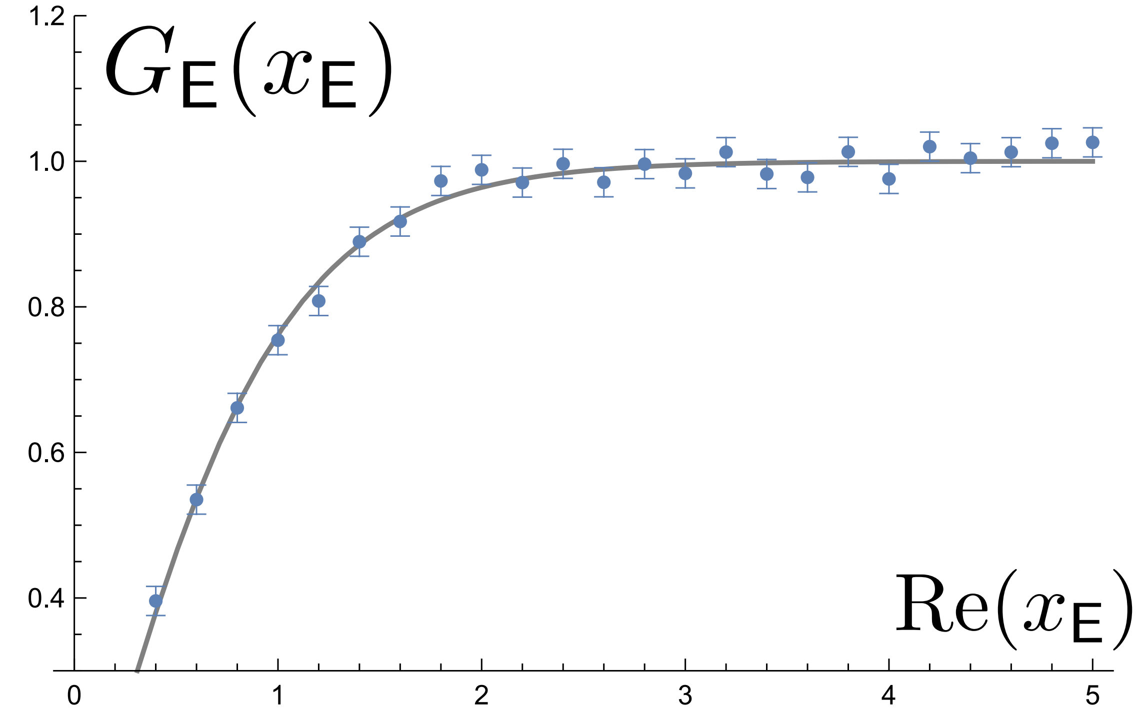


Limited numerical knowledge

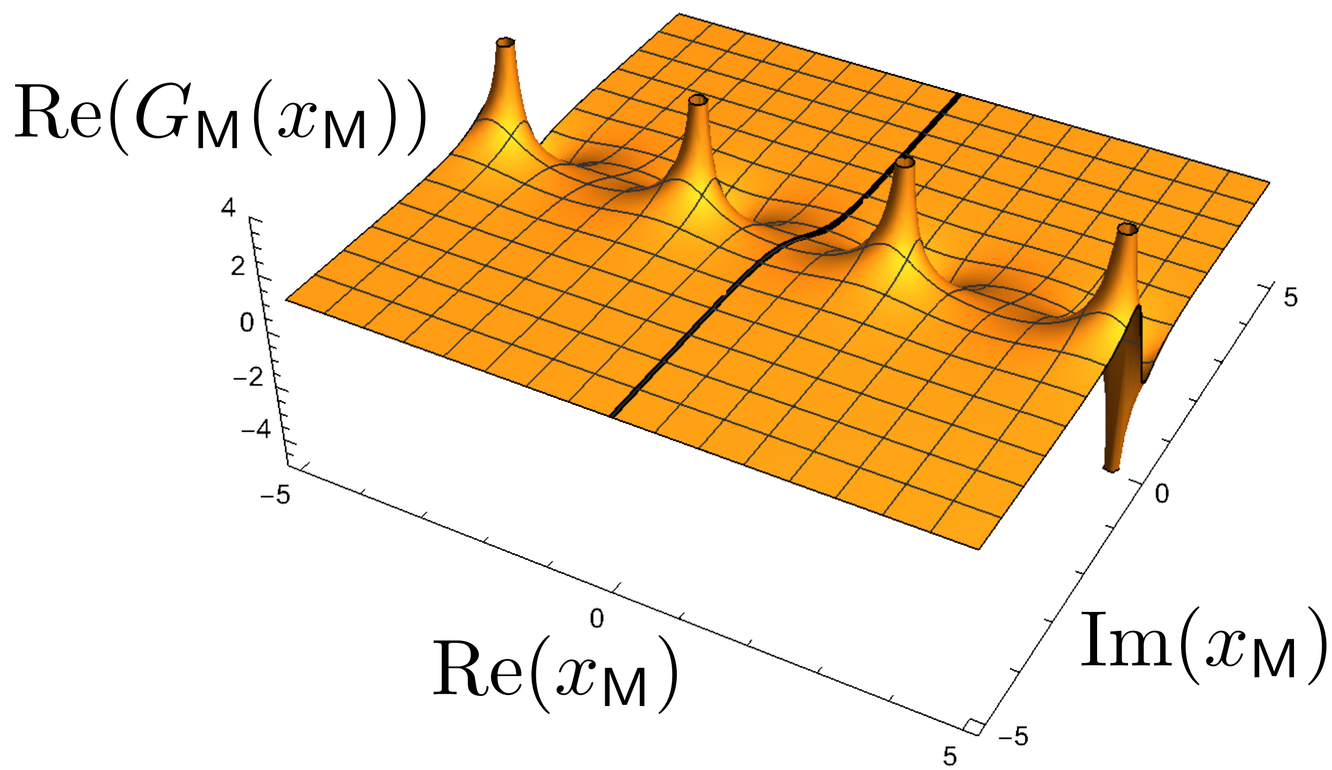
Suppose the Euclidean correlator is

$$G_{\mathsf{E}}(x_{\mathsf{E}}) = \tanh(x_{\mathsf{E}})$$

and we have a numerical estimate



Now the analytic continuation is ill-conditioned



Meaning of Euclidean

When asked **"Is it Euclidean or Minkowski?"** first think:

"Is the question well-posed?"

Sometimes it is

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Often it is not

- Masses
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Mostly the lectures are concerned with these quantities

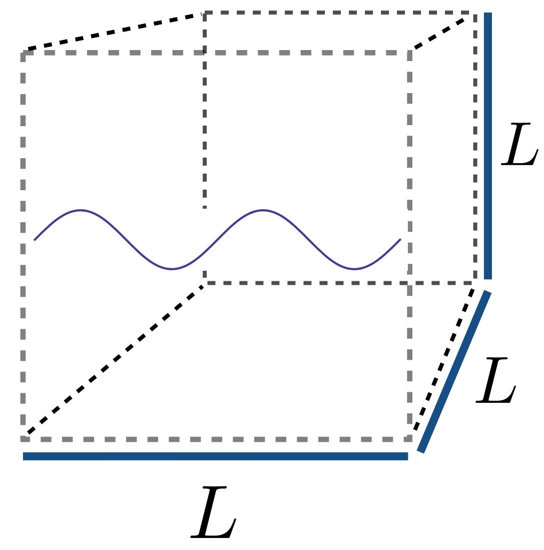
Correlators (where M vs. E is meaningful) are often used to access observables (where M vs. E is not meaningful)

Finite-volume setup

cubic, spatial volume (extent L)

periodic boundary conditions*

$$q(\tau,\mathbf{x}) = q(\tau,\mathbf{x}+L\mathbf{e}_i) \quad\Big|\quad \pi(\tau,\mathbf{x}) = \pi(\tau,\mathbf{x}+L\mathbf{e}_i)$$



time direction infinite*

continuum theory

*will also briefly consider finite T effects, alternative boundary conditions

$$\pi(\tau,\mathbf{x}) = \int_L d^3\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\,\tilde{\pi}(\tau,\mathbf{p})$$

$$\pi(\tau,\mathbf{x}+L\mathbf{e}_i) = \int_L d^3\mathbf{x}\, e^{-i\mathbf{p}\cdot(\mathbf{x}+L\mathbf{e}_i)}\,\tilde{\pi}(\tau,\mathbf{p})$$

these must be equal

$$e^{-i\mathbf{p}\cdot L\mathbf{e}_i} = 1 \quad\Longrightarrow\quad p_i L = 2\pi n_i$$

Quantization of momentum

$$\mathbf{p} = \frac{2\pi}{L}\mathbf{n}\,, \qquad \mathbf{n}\in\mathbb{Z}^3$$

Fourier transform conventions (here in one-dimension)

Begin in infinite volume

$$\mathcal{FT}^{-1}\Big[\mathcal{FT}\big[f\big]\Big] = f \quad \longrightarrow \quad \int \frac{dp}{N}\, e^{ipx} \left[\int \frac{dx'}{N'}\, e^{-ipx'} f(x') \right] \stackrel{!}{=} f(x)$$

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$$= 1$$

simplify via...

$$\int dp\, e^{ip(x-x')} = 2\pi\delta(x-x')$$

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Now repeat in a finite volume

$$\frac{1}{N}\sum_n e^{ix(2\pi n/L)}\Big[\int_0^L dx'\, e^{-ix'(2\pi n/L)} f(x')\Big]$$

Rewrite this using $\displaystyle\sum_n e^{2\pi i z} = \sum_{n'} \delta(z + n')$

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$$\frac{1}{N}\sum_n e^{ix(2\pi n/L)}\Big[\int_0^L dx'\, e^{-ix'(2\pi n/L)} f(x')\Big] = \int_0^L dx' \sum_{n'} \frac{\delta\big[(x'-x)/L - n'\big]}{N} f(x') = \frac{L f(x)}{N}$$

$$= 1$$

Rewrite this using $\sum_n e^{2\pi i z} = \sum_{n'} \delta(z + n')$

argument can only vanish when $n' = 0$

Fourier transform conventions (here in one-dimension)

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Now repeat in a finite volume

$$\mathcal{FT}_L^{-1}\Big[\mathcal{FT}_L[f]\Big] = f \quad \longrightarrow \quad \frac{1}{L} \sum_p e^{ipx} \left[\int_L dx'\, e^{-ipx'} f(x') \right] = f(x)$$

$p = 2\pi n / L$

Fourier transform conventions (here in one-dimension)

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Take the infinite-volume limit as a sanity check

$$\lim_{L\to\infty}\frac{1}{L}\sum_p f(p) = \lim_{L\to\infty}\frac{1}{L}\sum_n f(2\pi n/L) = \lim_{L\to\infty}\frac{1}{L}\int dn\, f(2\pi n/L) = \lim_{L\to\infty}\frac{1}{L}\int dp \frac{L}{2\pi}\, f(p) = \int \frac{dp}{2\pi} f(p)$$

For a smooth function we can
replace the sum with an integral

Fourier transform conventions (here in one-dimension)

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$$\lim_{L\to\infty} \left[\frac{1}{L}\sum_p - \int \frac{dp}{2\pi} \right] f(p) = 0$$

Feynman rules

Infinite-volume scalar propagator

$$(-\partial^2 + M^2)\, G(x-x') = \delta^4(x-x') \quad + \text{ b.c.} \quad \Longrightarrow \quad G(x-x') = \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot(x-x')} \frac{1}{p^2+M^2-i\epsilon}$$

Finite-volume scalar propagator

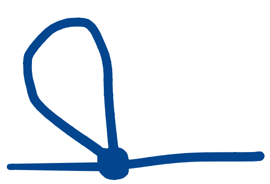
$$(-\partial^2 + M^2)\, G_L(x-x') = \sum_{\vec{n}} \delta^4(x-x'-\vec{n}L) \quad + \text{ b.c.} \quad \longrightarrow \quad G_L(x-x') = \int \frac{dp^0}{2\pi} \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_{\vec{n}} e^{-i\vec{p}\cdot\vec{n}L}\, \frac{e^{ip\cdot(x-x')}}{p^2+M^2-i\epsilon}$$

$$\sum_{\vec{n}} e^{-i\vec{p}\cdot\vec{n}L} = \sum_{\vec{n}'} (2\pi)^3 / L^3\, \delta^3\!\left(\vec{p} - \frac{2\pi\vec{n}'}{L}\right)$$

$$\boxed{G_L(x-x') = \frac{1}{L^3} \sum_{\vec{p}} \int \frac{dp^0}{2\pi} \frac{e^{ip\cdot(x-x')}}{p^2+M^2-i\epsilon}}$$

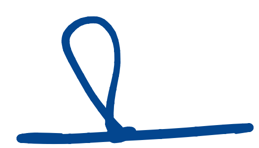
Key difference in Feynman rules

Infinite-volume:



$$= \int \frac{d^4k}{(2\pi)^4} \cdots$$

Finite-volume:



$$= \frac{1}{L^3} \sum_{\vec{k}} \int \frac{dk^0}{2\pi}$$

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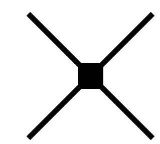
  - ☐ Different formalisms
  - ☐ Basic examples

Particle mass in $\lambda\phi^4$

- Definition of the theory (bare parameters, Minkowski)

$$\mathcal{L}(x) = -\frac{1}{2}\phi_0(x)(-\partial^2 + m_0^2)\phi_0(x) - \frac{\lambda_0}{4!}\phi_0(x)^4$$

 $= \frac{1}{i}\frac{1}{p^2+m_0^2}$

 $= -i\lambda_0$

- Infinite-volume mass

$$\tilde{G}(p) = \int d^4x\, e^{-ip\cdot x}\langle\phi(x)\phi(0)\rangle = \frac{1}{i}\frac{|Z|^2}{p^2+M^2} + O[(p^2+M^2)^0]$$

also $|Z|^2 = 1 + O(\lambda_0^2)$ + $M^2 = m_0^2 + O(\lambda_0)$

$$\longrightarrow \tilde{G}(p) = \frac{1}{i}\frac{1}{p^2+m_0^2} + \left(\frac{1}{i}\frac{1}{p^2+m_0^2}\right)^2(-i\,\delta^{(1)}M^2) + O(\lambda_0^2)$$

also $\tilde{G}(p) =$  $+ \cdots$

$$\longrightarrow -i\,\delta^{(1)}M^2 = \frac{1}{2}\frac{(-i\lambda_0)}{i}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+M^2} = \frac{\lambda_0}{2}\int\frac{d^3k}{(2\pi)^3}\int\frac{dk^0}{2\pi}\frac{1}{(k^0)^2-\omega_k^2+i\varepsilon}$$

$$\omega_k = \sqrt{\vec{k}^2+M^2}$$

$$\longrightarrow \boxed{\delta^{(1)}M^2 = \frac{\lambda_0}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k} = \frac{\lambda_0}{2}\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+M^2}}$$

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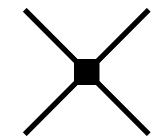
$$\mathcal{L}(x) = -\frac{1}{2}\phi_0(x)(-\partial^2 + m_0^2)\phi_0(x) - \frac{\lambda_0}{4!}\phi_0(x)^4$$

$= \frac{1}{i}\frac{1}{p^2+M_0^2}$



$= -i\lambda_0$

- Finite-volume mass correction

$$\tilde{G}_L(p_0) = \int d^3x \int dx^0\, e^{+ip_0x_0} \langle \phi(x)\phi(0)\rangle_L = \frac{1}{i}\frac{|Z|^2}{-(p_0)^2+M(L)^2} + \mathcal{O}[(\ldots)^0]$$

$$= \frac{1}{i}\frac{1}{-p_0^2+M_0^2} + (\ldots)^2(-i\,\delta^{(\lambda)}M^2(L)) =$$

$+$

$+ \cdots \longrightarrow \delta^{(\lambda)}M^2(L) = \frac{\lambda_0}{2}\frac{1}{L^3}\sum_{\vec{k}}\int\frac{dk_4}{2\pi}\frac{1}{k^2+M^2}$

$$\Delta M^2(L) \equiv M(L)^2 - M^2 = +\frac{\lambda_0}{2}\left[\frac{1}{L^3}\sum_{\vec{k}} - \int_{\vec{k}}\right]\int_{k_4}\frac{1}{k^2+M^2} = +\frac{\lambda_0}{2}\sum_{\hat{n}\neq 0}\int\frac{d^4k}{(2\pi)^4}\frac{e^{i\vec{n}\cdot\vec{k}L}}{k^2+M^2}$$

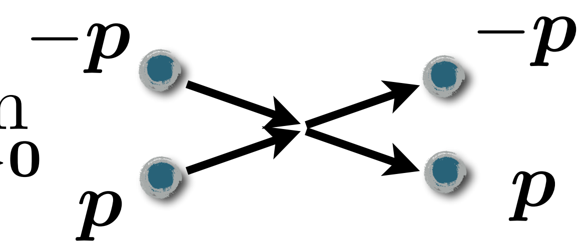
$$\Delta M^2(L) = \frac{\lambda_0}{2}\sum_{\vec{n}\neq 0} g(\vec{n}^2M^2L^2) \quad \text{where} \quad g(x^2) = \frac{K_1(x)}{4\pi x}$$

Particle mass in $\lambda\phi^4$

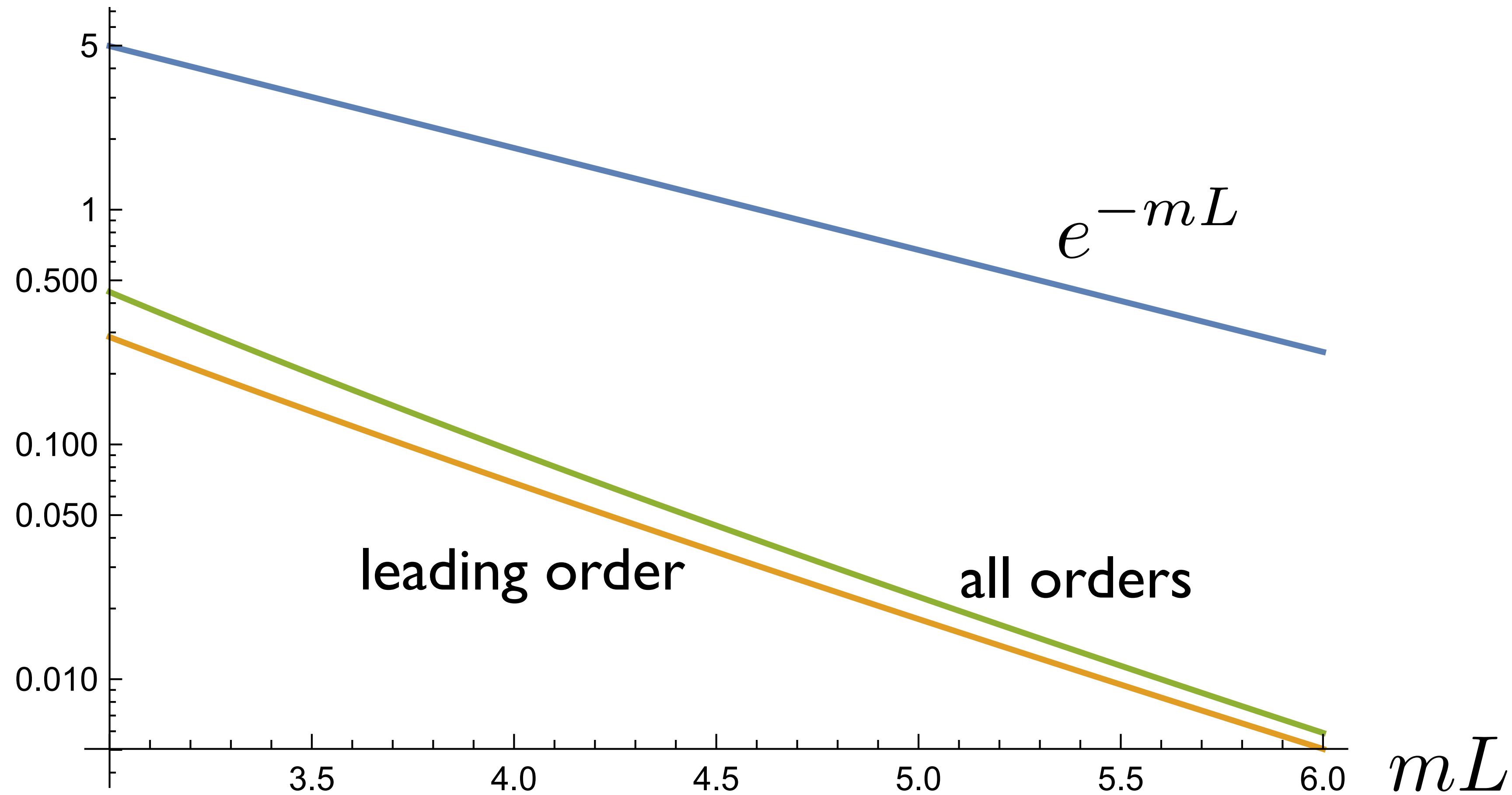
- Leading order volume correction ($\Delta m^2(L) = m(L)^2 - m^2 > 0$)

$$\Delta m^2(L) = \frac{\lambda_0}{2} \sum_{\boldsymbol{n} \neq \boldsymbol{0}} \frac{K_1(|\boldsymbol{n}|mL)}{4\pi|\boldsymbol{n}|mL} = \frac{3\lambda_0}{4\sqrt{2\pi}} \frac{e^{-mL}}{(mL)^{3/2}} \left[1 + \mathcal{O}(1/L, e^{-\beta mL})\right]$$

- To the order we work: coupling = threshold scattering amplitude $\lim_{\boldsymbol{p}\to\boldsymbol{0}}$

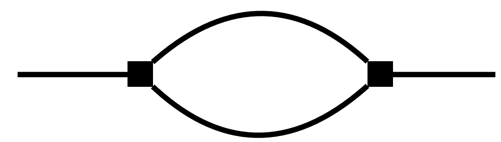


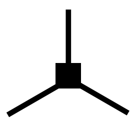
$\Delta m^2(L)/m^2 \times 100$



Particle mass in $g\phi^3$

$$\Delta_{\boldsymbol{k}} \equiv \left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right]$$





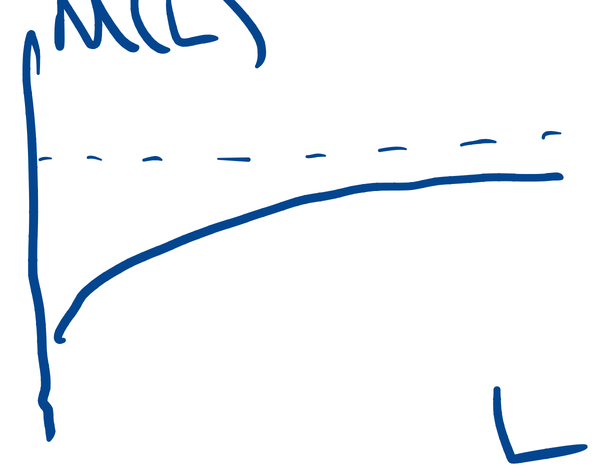
$$\Delta m^2(L) = i\frac{g^2}{2}\Delta_{\boldsymbol{k}} \int \frac{dk^0}{2\pi} \frac{1}{(k^0)^2 - \omega_{\boldsymbol{k}}^2 + i\epsilon} \frac{1}{(k^0 - m)^2 - \omega_{\boldsymbol{k}}^2 + i\epsilon}$$

$$\Rightarrow \Delta^2 M(L) = \frac{g^2}{2}\Delta_{\vec{k}} \frac{1}{2\omega_{\vec{k}}}\left[\frac{1}{(\omega_k - M)^2 - \omega_k^2} + \frac{1}{(\omega_k + M)^2 - \omega_k^2}\right]$$

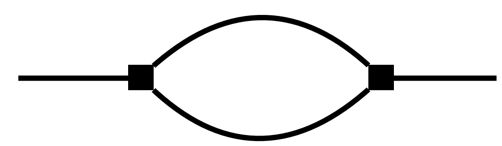
$$= \frac{g^2}{2M}\int \frac{d^3k}{(2\pi)^3} \sum_{\vec{n}} e^{i\vec{n}L\cdot\vec{k}} \frac{1}{2\omega_k} \frac{-1}{\vec{k}^2 + 3/4 M^2}$$

$$= -6\,\frac{g^2}{4}\,\frac{2\pi}{8\pi^3}\int_0^\infty dk\, k^2 \underbrace{\int_{-1}^{1} dc\, e^{iLkc}}_{\frac{2\,\mathrm{Im}[e^{iLk}]}{Lk}} \frac{1}{2\omega_k} \frac{1}{(k - i\sqrt{3/4}M)} \frac{1}{(k + i\sqrt{3/4}M)}$$

$$= -6\,\frac{g^2}{16\pi M L} e^{-\sqrt{3/4}ML} + \mathcal{O}(e^{-ML})$$

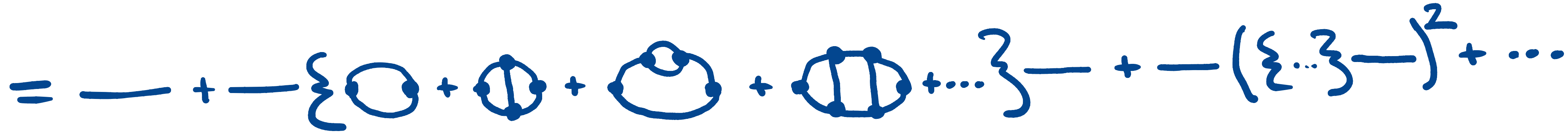


Matrix element in $g\phi^3$



$$\Delta_{\boldsymbol{k}} \equiv \left[\frac{1}{L^3} \sum_{\boldsymbol{k}} - \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \right]$$

$$\widetilde{G}_L(p^0) = \frac{1}{i} \frac{1}{-(p^0)^2 + m_0^2 + \Sigma_L(-p_0^2)}$$



$$= \frac{1}{i} \frac{1}{-(p^0)^2 + m^2_{(L)} + [-(p^0)^2 + m(L)^2]\Sigma'_L(-m^2)}$$

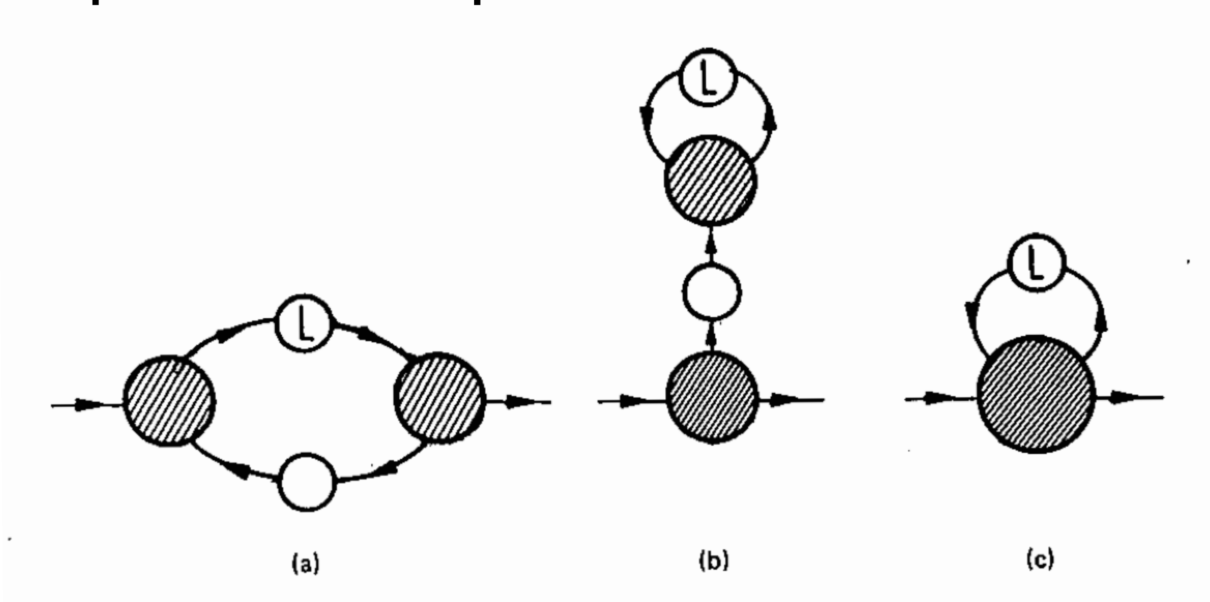
$$= \frac{1}{i} \left(\frac{1}{1+\Sigma'_L(-m^2)} \right) \cdot \frac{1}{-(p^0)^2 + m(L)^2} = \frac{1}{i} \frac{|Z(L)|^2}{-(p^0)^2 + m(L)^2}$$

$$\left(\Sigma_L(-(p^0)^2) = i\frac{g^2}{2} \Delta_{\boldsymbol{k}} \int \frac{dk^0}{2\pi} \frac{1}{(k^0)^2 - \omega_{\boldsymbol{k}}^2 + i\epsilon} \frac{1}{(k^0 - p^0)^2 - \omega_{\boldsymbol{k}}^2 + i\epsilon} \right)$$

$$\frac{|\langle 0|\phi(0)|\phi, L\rangle|^2}{|\langle 0|\phi(0)|\phi, \infty\rangle|^2} = \frac{1 + \Sigma'_\infty(-m^2)}{1 + \Sigma'_L(-m^2)} = 1 - \left[\Sigma'_L(-m^2) - \Sigma'_\infty(-m^2) \right] + \cdots$$

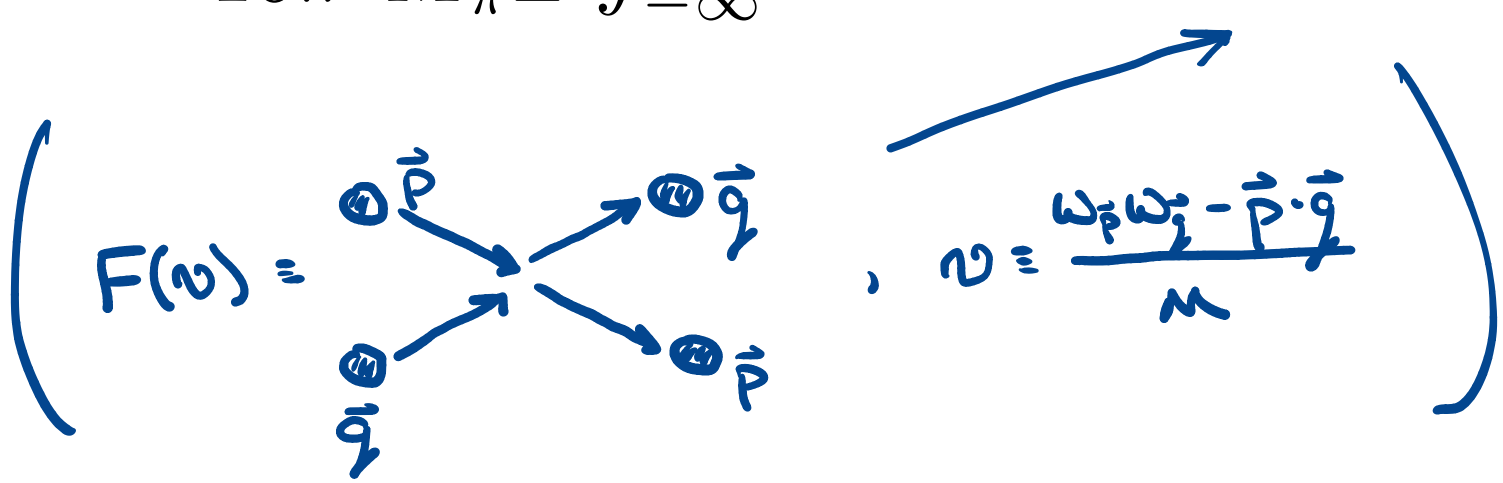
Pion mass

- Seminal work by M. Lüscher (1986)
- All orders skeleton expansion to express result in terms of irreducible vertices



- Result

$$\Delta M_\pi(L) = \frac{3}{16\pi^2 M_\pi L}\int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}\,L}\,[-F(iy)] + \cdots$$



Lüscher (1986)

Five-day outline

- [x] Warm-up and definitions
  - [x] Meaning of Euclidean
  - [x] Finite-volume set-up
- [ ] e^{-mL} volume effects
  - [x] Mass, matrix element in $\lambda\phi^4, g\phi^3$
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  - [ ] LO-HVP for $(g-2)_\mu$
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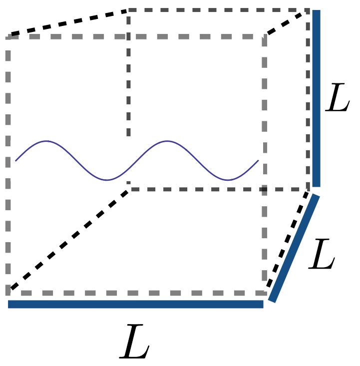
  - [ ] Different formalisms
  - [ ] Basic examples

$(g-2)_\mu$ LO HVP (hadronic vacuum polarization)

- $T \times L^3$ periodic, Euclidean signature



$$a_\mu^{\mathrm{LO,HVP}}(L,T) \equiv \int_0^{T/2} dx_0\, \mathcal{K}(x_0) \int_{L^3} d^3\boldsymbol{x}\, \langle j_k^\dagger(x_0,\boldsymbol{x}) j_k(0) \rangle_{L,T}$$

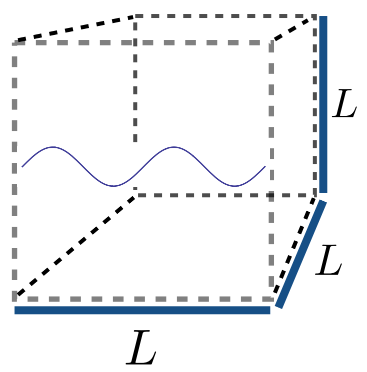
- Continuous kernel function: $\mathcal{K}(x_0) \overset{x_0 \to \infty}{\propto} x_0^2$
- For asymptotically large T, L …

$$\begin{aligned} a_\mu^{\mathrm{LO,HVP}}(L,T) - a_\mu^{\mathrm{LO,HVP}} &= \mathcal{O}(e^{-M_\pi L}) + \mathcal{O}(e^{-\sqrt{2} M_\pi L}) + \mathcal{O}(e^{-\sqrt{3} M_\pi L}) + \mathcal{O}(e^{-1.9 M_\pi L}) + \cdots \\ &+ \mathcal{O}(e^{-M_\pi T}) + \mathcal{O}(e^{-\frac{3}{2} M_\pi T}) + \cdots \\ &+ \mathcal{O}(e^{-M_\pi \sqrt{T^2+L^2}}) + \cdots \\ &+ \mathcal{O}(e^{-M_K L}) + \cdots \end{aligned}$$

$\sqrt{2+\sqrt{3}} \approx 1.92$

$(g-2)_\mu$ LO HVP (hadronic vacuum polarization)

- *T x L*³ periodic, Euclidean signature



$$a_\mu^{\mathrm{LO,HVP}}(L,T) \equiv \int_0^{T/2} dx_0\, \mathcal{K}(x_0) \int_{L^3} d^3\boldsymbol{x}\, \langle j_k^\dagger(x_0,\boldsymbol{x}) j_k(0)\rangle_{L,T}$$

- Continuous kernel function: $\mathcal{K}(x_0) \overset{x_0\to\infty}{\propto} x_0^2$
- For asymptotically large *T, L* …

$$\begin{aligned} a_\mu^{\mathrm{LO,HVP}}(L,T) - a_\mu^{\mathrm{LO,HVP}} = & \; \mathcal{O}(e^{-M_\pi L}) + \mathcal{O}(e^{-\sqrt{2}M_\pi L}) + \mathcal{O}(e^{-\sqrt{3}M_\pi L}) + \mathcal{O}(e^{-1.9 M_\pi L}) + \cdots \\ & + \mathcal{O}(e^{-M_\pi T}) + \mathcal{O}(e^{-\frac{3}{2}M_\pi T}) + \cdots \\ & + \mathcal{O}(e^{-M_\pi \sqrt{T^2+L^2}}) + \cdots \\ & + \mathcal{O}(e^{-M_K L}) + \cdots \end{aligned}$$

$\sqrt{2+\sqrt{3}} \approx 1.92$

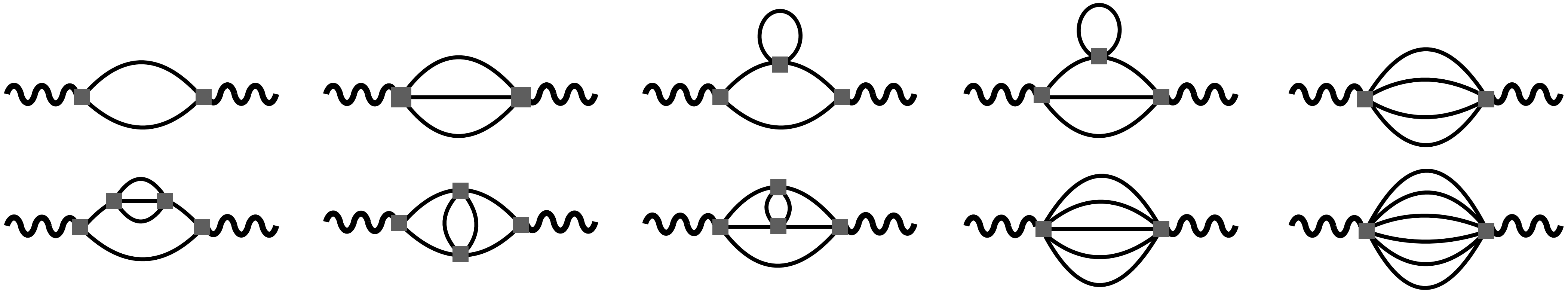
1904.10010   2004.03935   for now

MTH and A. Patella • *strong inspiration from…* Lüscher 1986

Diagrammatic expansion

- All orders diagrammatic expansion

$$a_\mu(L) =$$



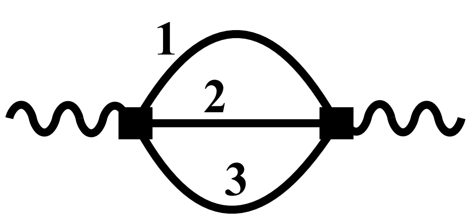
$$+\cdots$$

- Loop momenta are summed… rewrite using ***Poisson summation***

$$\frac{1}{L^3}\sum_{\boldsymbol{k}} \longrightarrow \sum_{\boldsymbol{n}} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{iL\boldsymbol{n}\cdot\boldsymbol{k}}$$

$\boldsymbol{n} = (n_x, n_y, n_z)$ can be interpreted as loop pion wrapping the torus

- Some subtleties in assigning ***Poisson modes***



$$\begin{matrix} \boldsymbol{n}_1 = (0,0,1) \\ \boldsymbol{n}_2 = (0,0,1) \end{matrix} \equiv \begin{matrix} \boldsymbol{n}_2 = (0,0,0) \\ \boldsymbol{n}_3 = (0,0,1) \end{matrix}$$

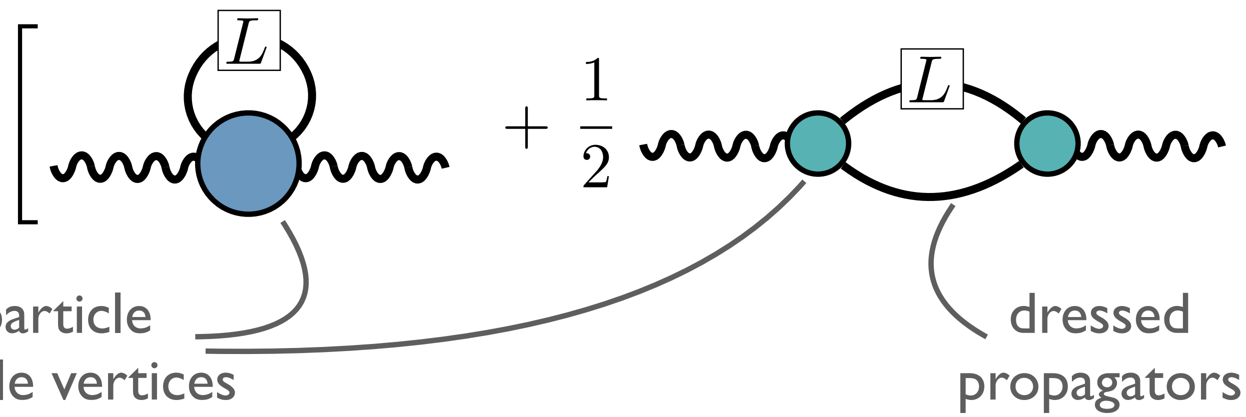
understood as a gauge redundancy → gauge fixing to catalogue effects

Lüscher (1986)

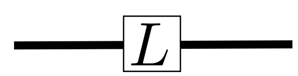
Leading terms

- Leading contributions: *only one pion wraps the torus*

$$\Delta a_\mu(L) = \int_0^\infty dx_0\, \mathcal{K}(x_0) \Big[\quad + \frac{1}{2} \quad \Big] + \mathcal{O}(e^{-1.9 M_\pi L})$$

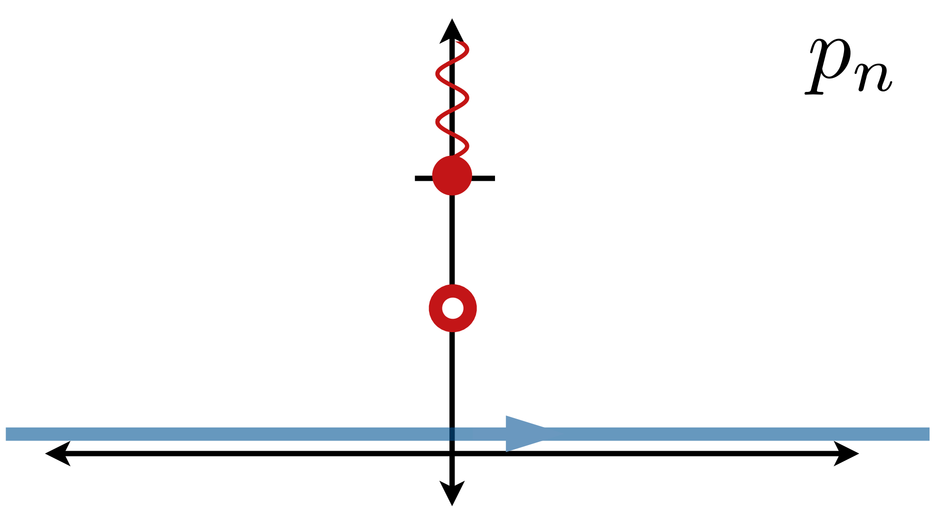


- All L dependence inside…

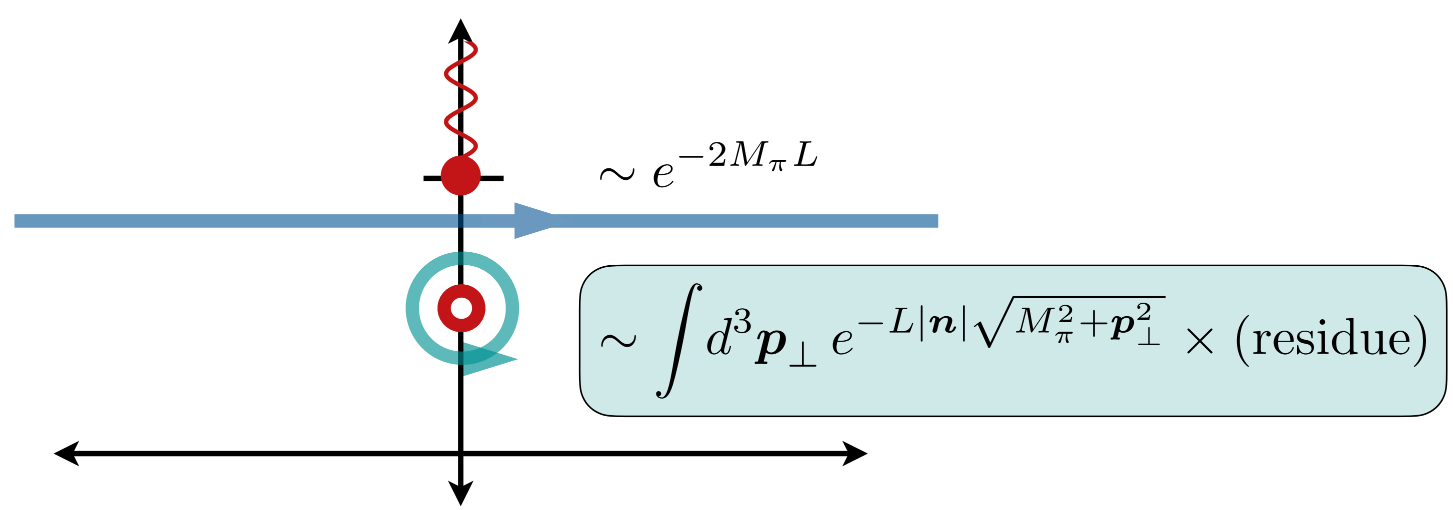


$$\equiv \sum_{\boldsymbol{n}} \frac{e^{iL\boldsymbol{n}\cdot\mathbf{p}}}{p^2 + M_\pi^2 + \Sigma(p^2)}$$

- Exponential decay? Decompose: $\mathbf{p} = p_n \hat{\boldsymbol{n}} + \mathbf{p}_\perp$



$\equiv$



pole contributions → on-shell pions → ***physical Compton amplitude***

Result

- Sum of ***all single-winding terms*** gives

$$\Delta a_\mu(L) = -\frac{\alpha^2}{m_\mu^2} \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \int \frac{dp_3}{2\pi} \frac{e^{-|\boldsymbol{n}|L\sqrt{M_\pi^2+p_3^2}}}{2\pi L|\boldsymbol{n}|} \int_0^\infty dx_0\, \widehat{\mathcal{K}}(m_\mu x_0) \int \frac{dk_3}{2\pi} \cos(x_0 k_3)\, \mathrm{Re}\, T(-k_3^2, -k_3 p_3)$$

volume dependence — kernel — Compton amplitude

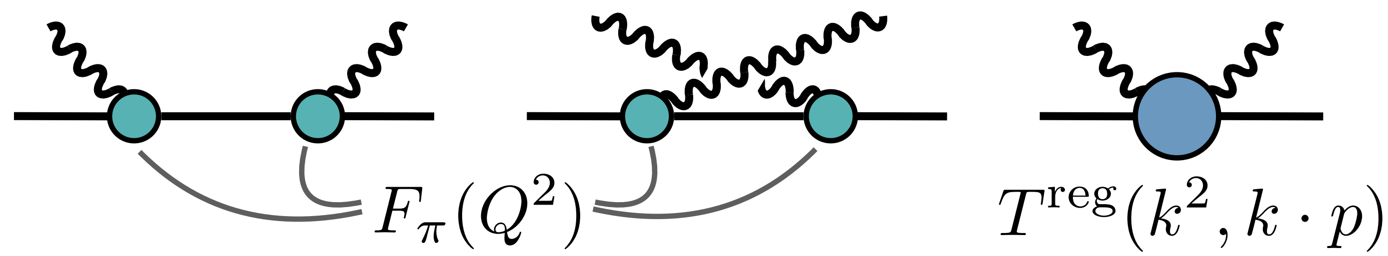
$$T(k^2, k\cdot p) \equiv i \lim_{\boldsymbol{p}'\to\boldsymbol{p}} \sum_{q=0,\pm1} \int d^4x\, e^{ikx} \langle \boldsymbol{p}', q|\mathrm{T}\mathcal{J}_\rho(x)\mathcal{J}^\rho(0)|\boldsymbol{p}, q\rangle$$

reference to EFT has disappeared

spacelike photons only

- Modeling the amplitude

$T(k^2, k\cdot p) =$



Monopole form

$$F_\pi(Q^2) = \frac{1}{1+Q^2/M^2}$$

One-loop ChPT

$$T^{\mathrm{reg}} = c_0 + c_1 \boldsymbol{k}^2 - \frac{7M_\pi^2 + 4\boldsymbol{k}^2}{6\pi^2 f_\pi^2}\, \sigma \cot^{-1}\sigma \Big|_{\sigma=\sqrt{1+4M_\pi^2/\boldsymbol{k}^2}}$$

D. Brommel et al. (QCDSF/UKQCD) (2007)

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