UK Research
and Innovation

QCD (and QED) in a Euclidean finite volume

EuroPLEx Summer School 2021

Maxwell T. Hansen

August 23rd-27th

THE UNIVERSITY
of EDINBURGH

Five-day outline

- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} volume effects
  - ☑ Mass, matrix element in $\lambda\phi^4, g\phi^3$
  - ☑ Pion mass
  - ☑ LO-HVP for $(g-2)_\mu$
  - ☐ Bethe-Salpeter kernel
- ☐ $2 \to 2$ finite-volume (f.v.) formalism
  - ☐ Scattering basics
  - ☐ Derivation
  - ☐ Generalizations
  - ☐ Applications
- ☐ $(1+)\mathscr{J} \to 2$ f.v. formalism
  - ☐ Derivation
  - ☐ Expansions
  - ☐ Applications
- ☐ $2 + \mathscr{J} \to 2$ f.v. formalism
  - ☐ Derivation
  - ☐ Testing the result
  - ☐ Numerical explorations
- ☐ Non-local matrix elements
  - ☐ Derivation
  - ☐ Applications
- ☐ $3 \to 3$ f.v. formalism
  - ☐ New complications
  - ☐ Derivation ($E_n(L)$ to $\mathscr{K}_{\text{df},3}$)
  - ☐ Integral equations ($\mathscr{K}_{\text{df},3}$ to $\mathscr{M}_3$)
  - ☐ Testing the result
  - ☐ Numerical explorations/calculations
- ☐ Finite-volume QED + QCD
  - ☐ Different formalisms
  - ☐ Basic examples

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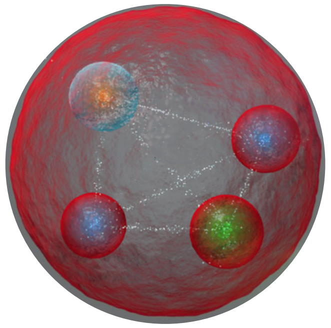
Multi-hadron observables

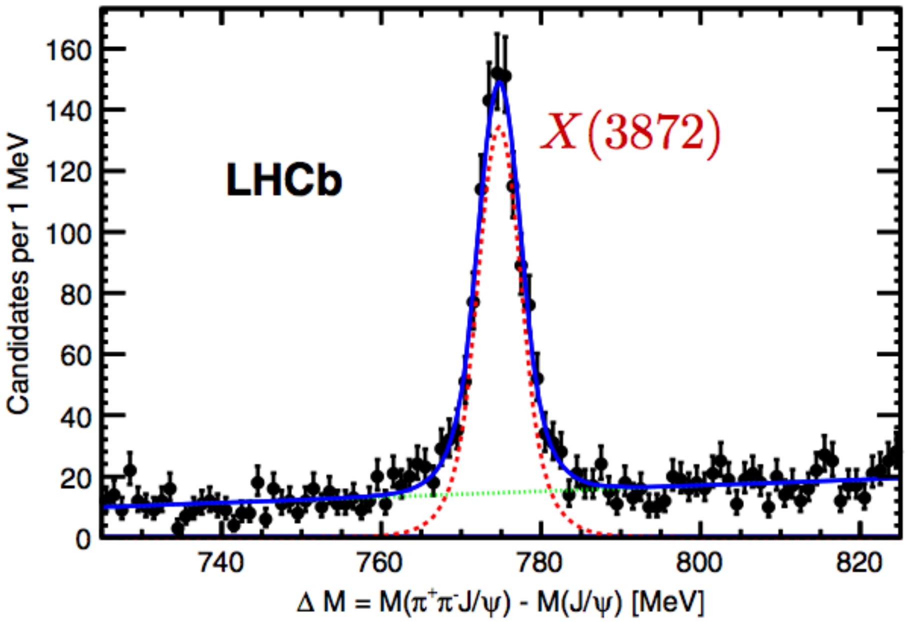
☐ Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$$\sim |D^0 \overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle\,?$$

• LHCb (PRD92, 2015) •



☐ Eletroweak, CP violation, resonant enhancement

CP violation in charm $D \to \pi\pi, K\overline{K}$

• LHCb (PRL, 2019) •

$f_0(1710)$ could enhance ΔA_{CP}

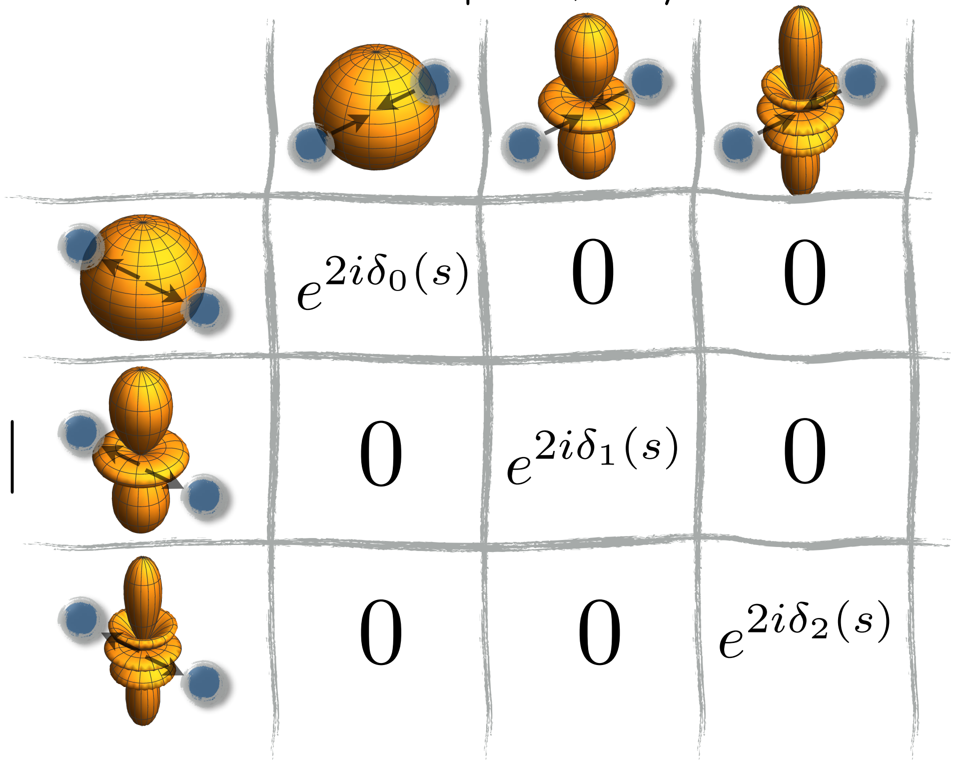
• Soni (2017) •

Resonant B decays $B \to K^*\,\ell\ell \to K\pi\,\ell\ell$

$|X\rangle, |\rho\rangle\,, |K^*\rangle\,, |f_0\rangle \quad \notin$ **QCD Fock space**

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- Overlaps of multi-hadron ***asymptotic states*** → S matrix



$$|\pi\pi, \text{in}\rangle$$

$$S(s) \equiv \langle \pi\pi, \text{out}| \quad \begin{array}{ccc} e^{2i\delta_0(s)} & 0 & 0 \\ 0 & e^{2i\delta_1(s)} & 0 \\ 0 & 0 & e^{2i\delta_2(s)} \end{array}$$

depends on $s = E_{\text{cm}}^2$
and angular variables

diagonal in angular momentum

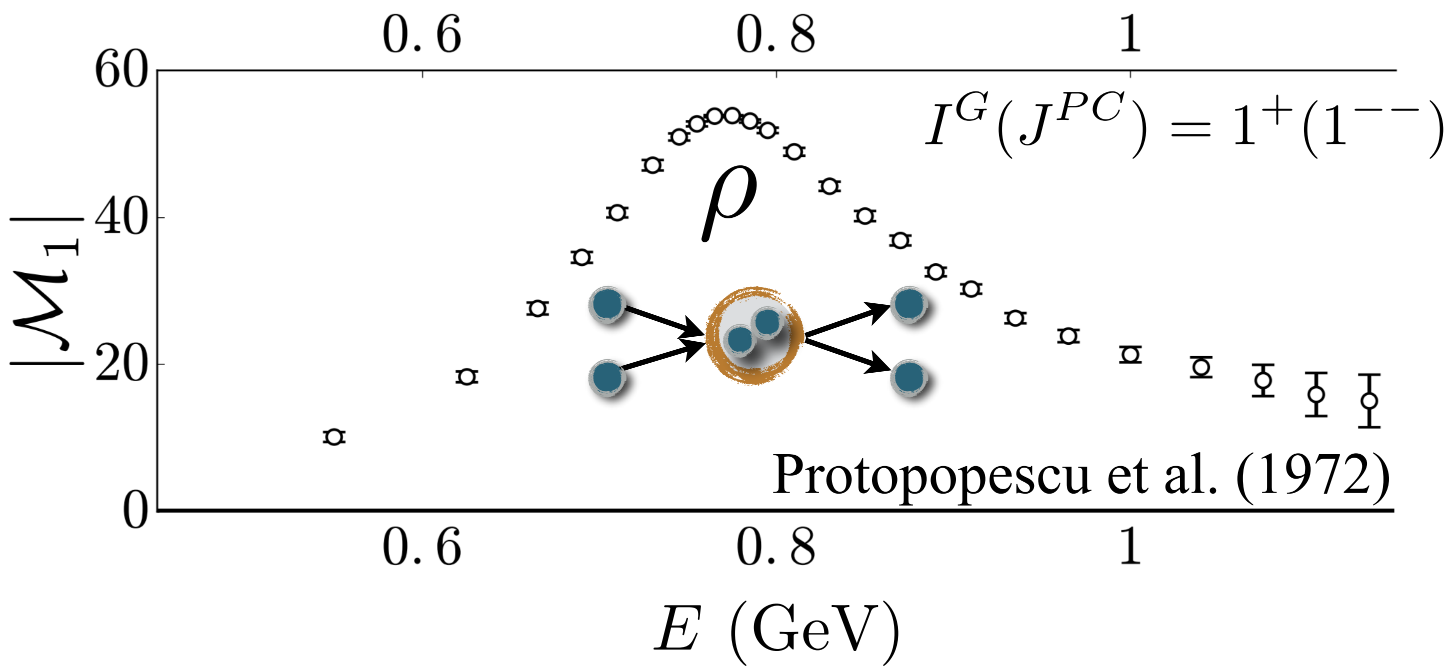
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

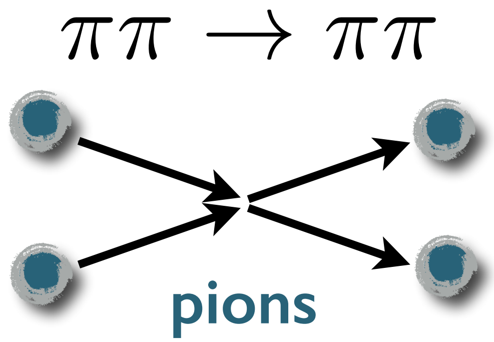
- ***An enormous space of information*** $|\pi\pi\pi\pi, \text{in}\rangle \quad |K\overline{K}, \text{in}\rangle \quad \cdots$

QCD resonances

- ☐ Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate





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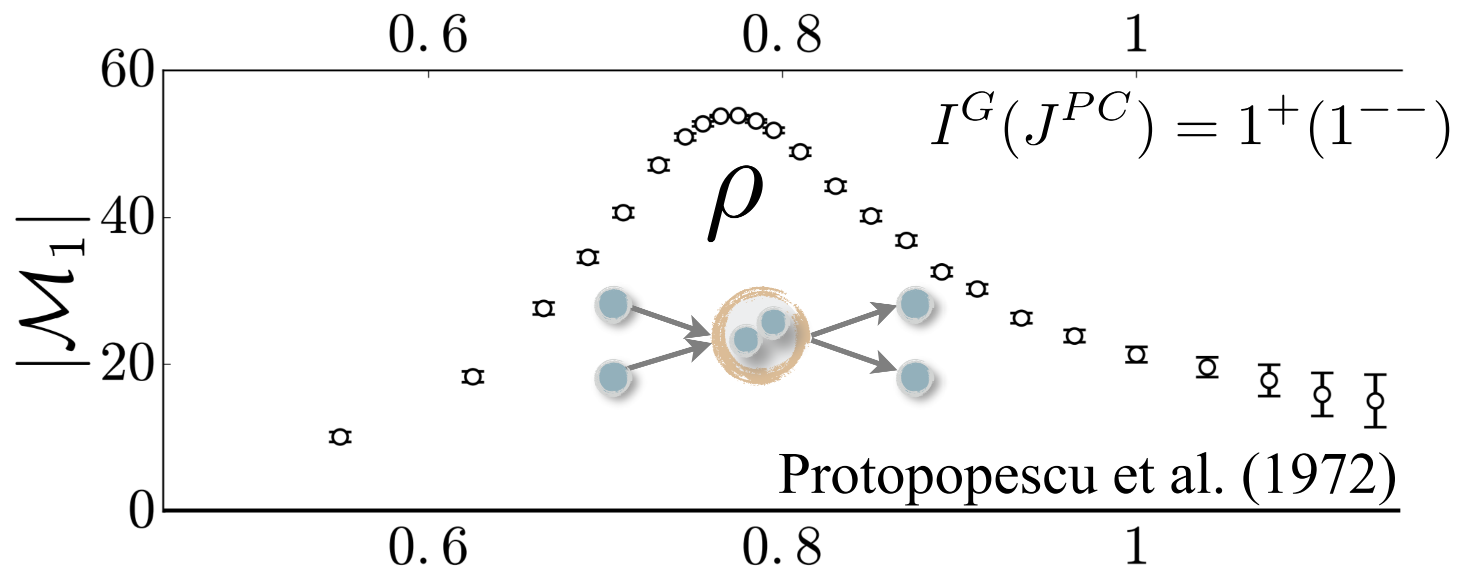
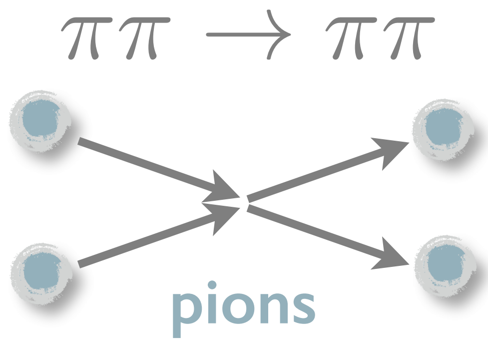
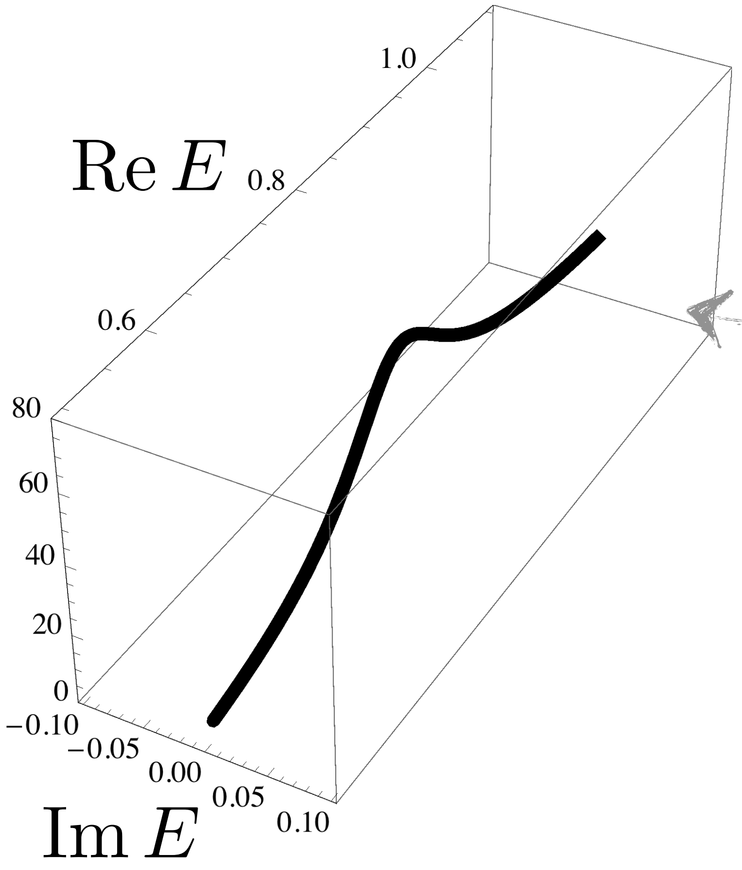


Protopopescu et al. (1972)



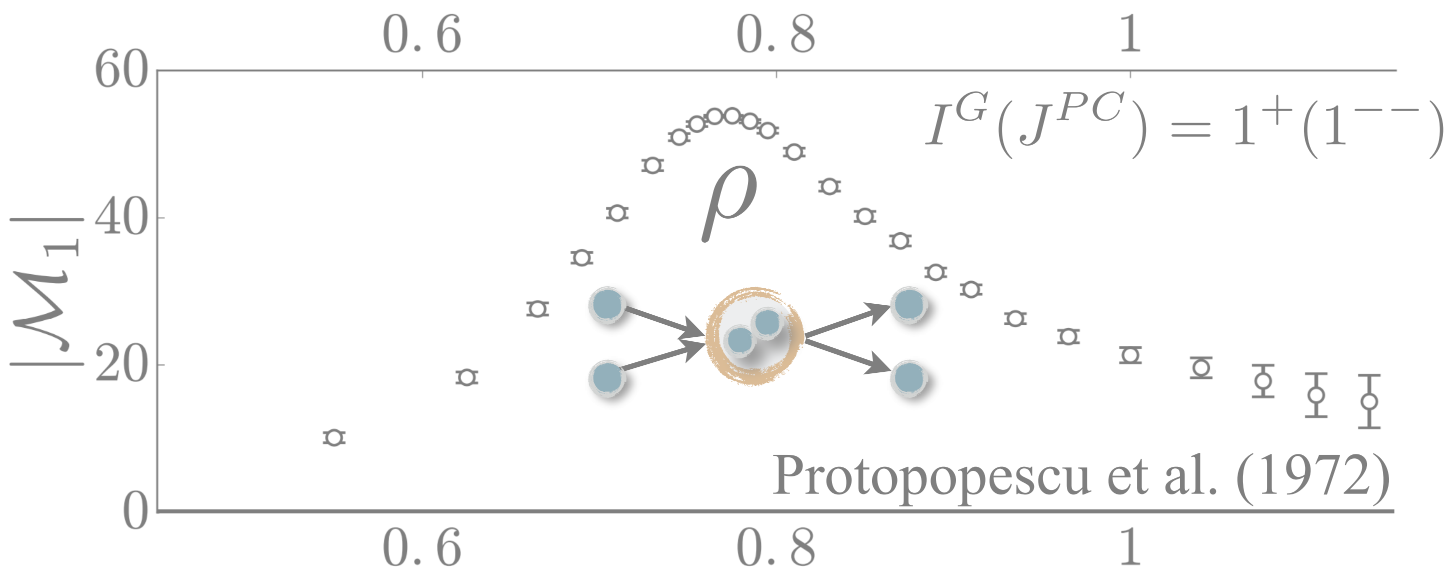


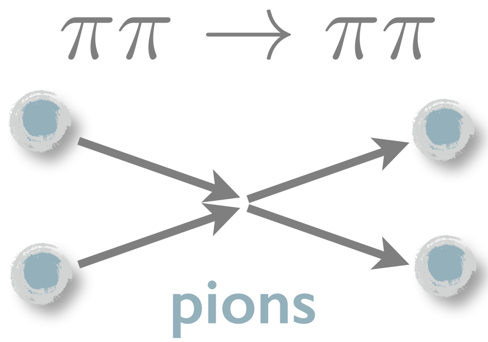
Extend to the **complex plane**

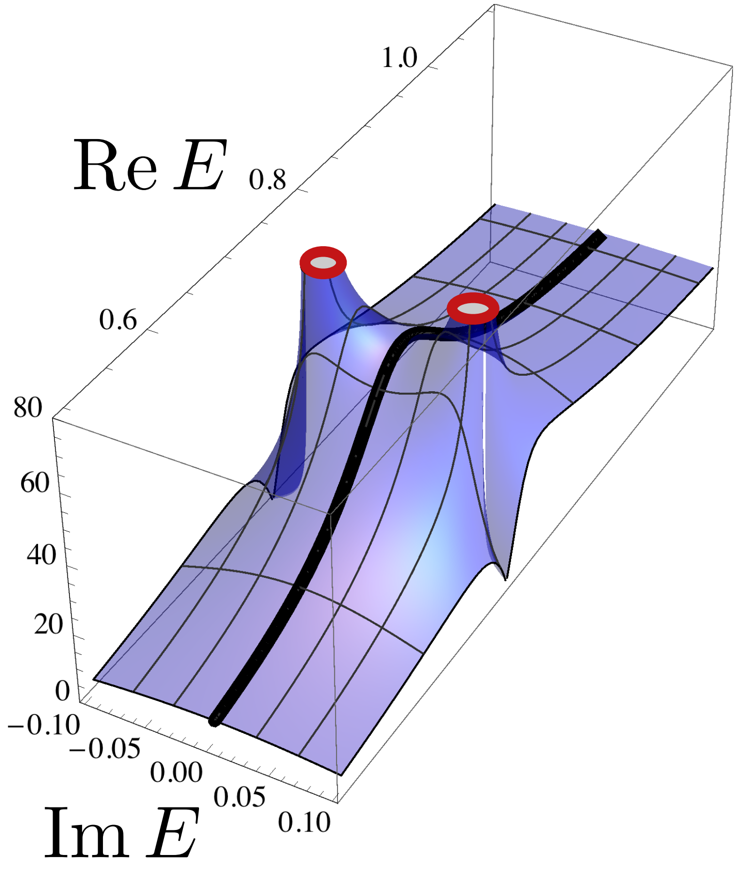
QCD resonances

☐ Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

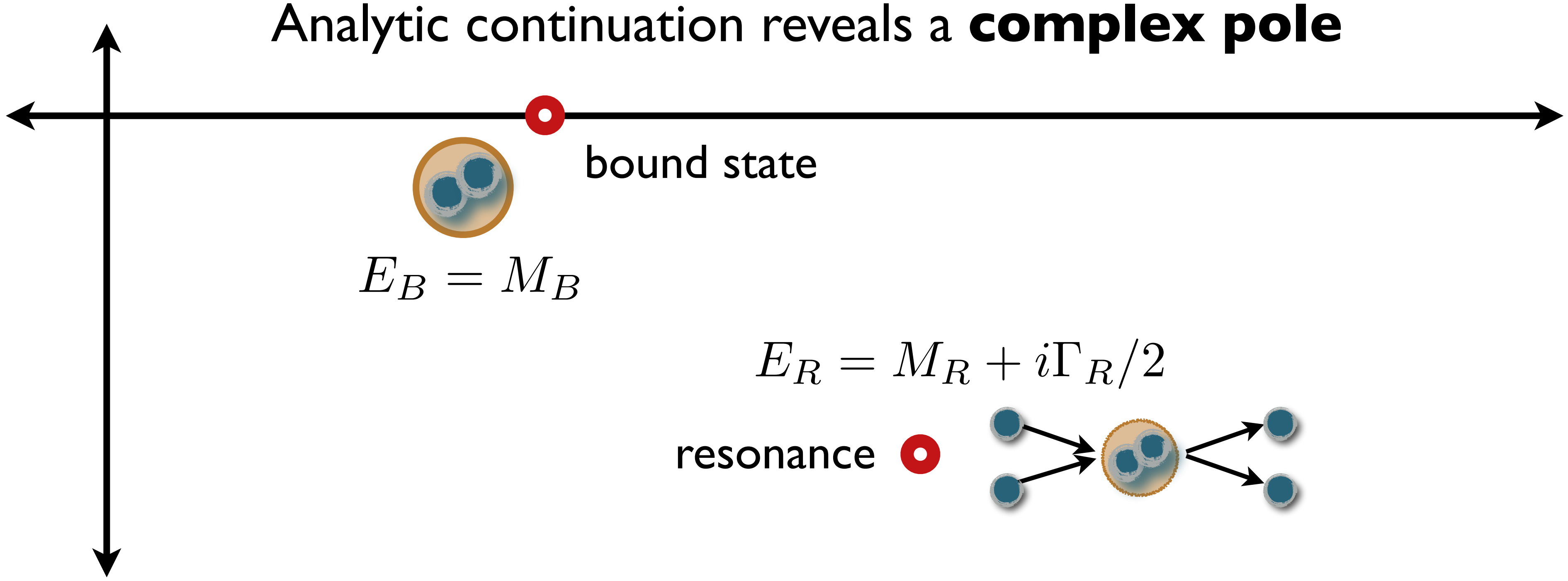
scattering rate







Analytic continuation reveals a **complex pole**



bound state

$E_B = M_B$

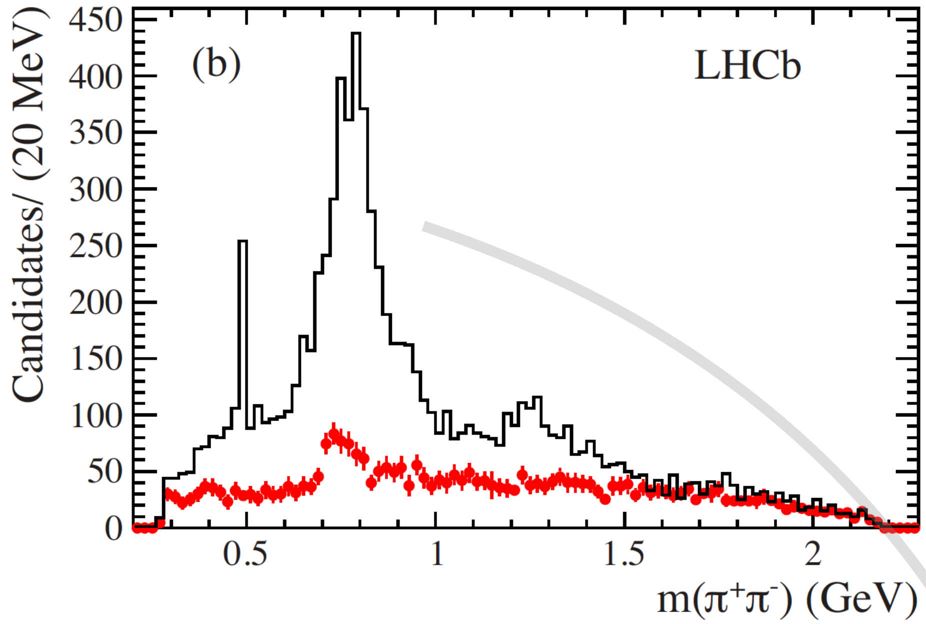
$E_R = M_R + i\Gamma_R/2$

resonance

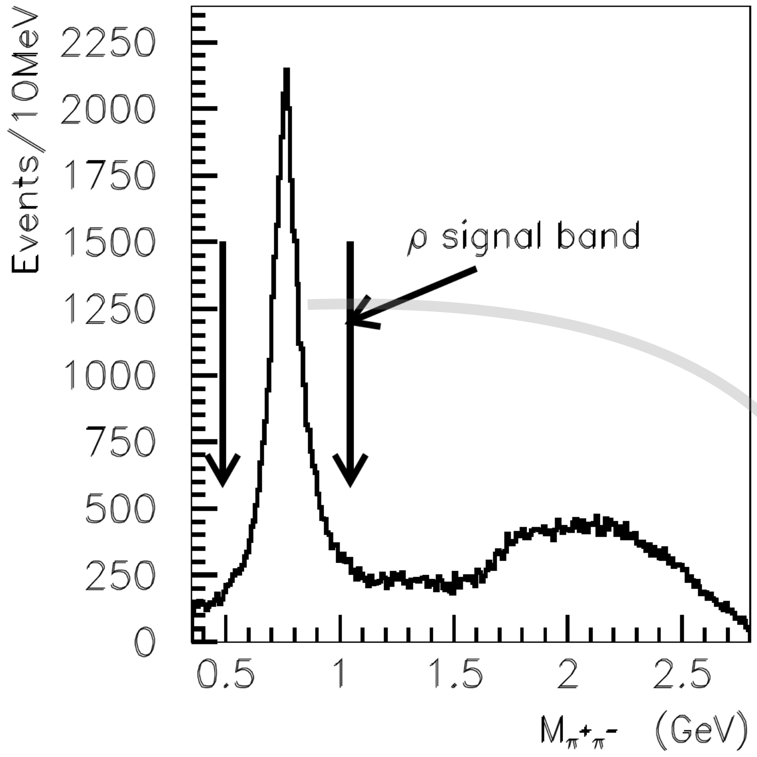
Pole is universal

☐ Resonances often seen in “production”

$\overline{B}^0 \to J/\psi\,\pi^+\pi^-$

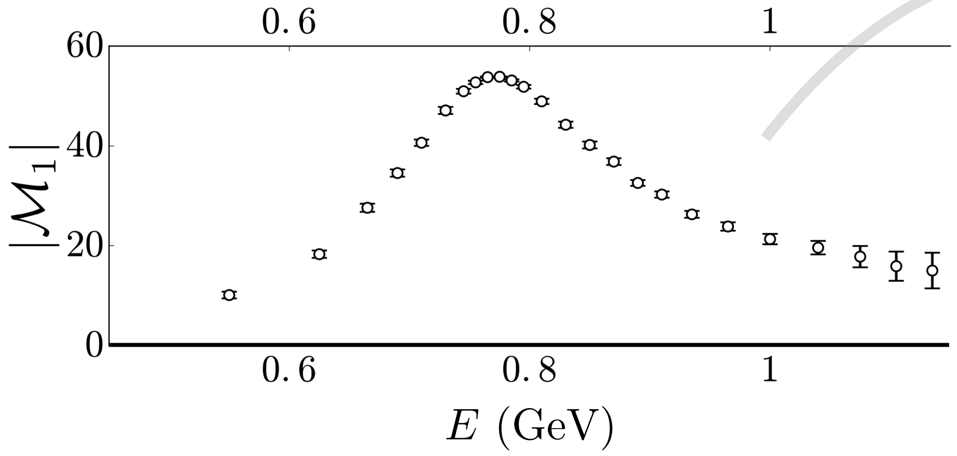


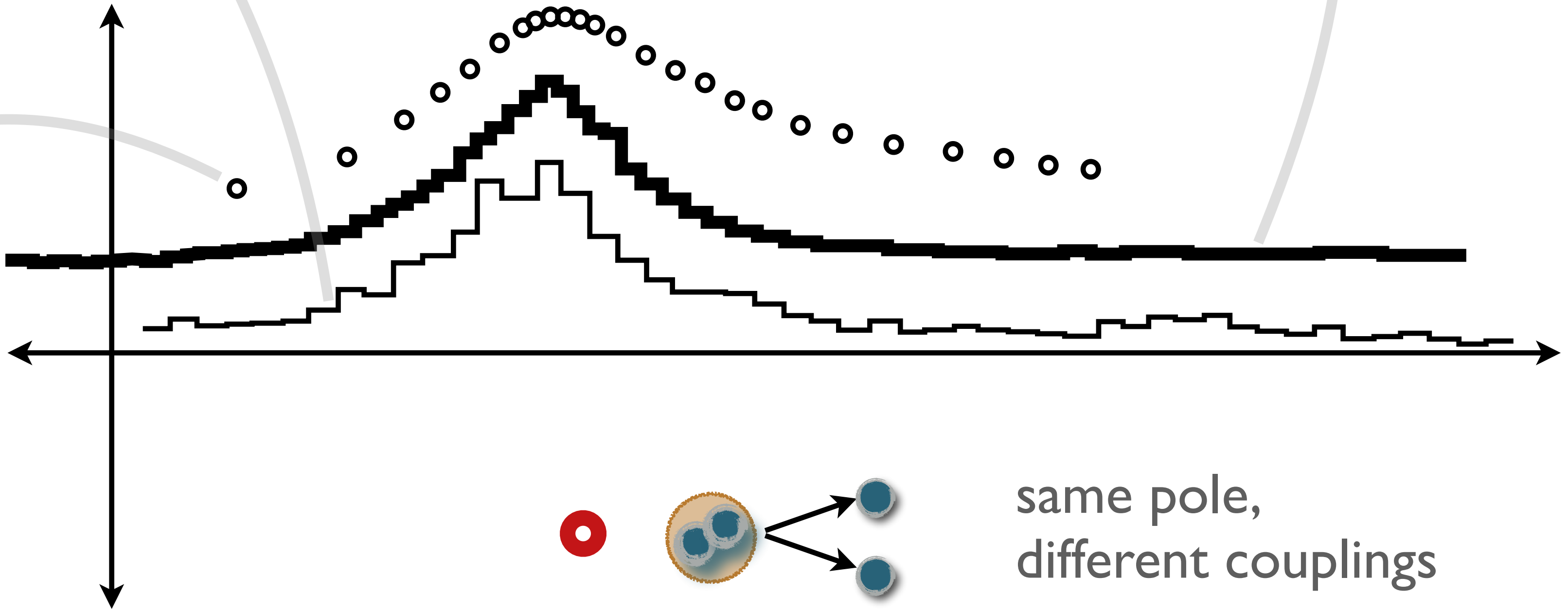
$J/\psi \to \gamma\gamma\rho$



(as opposed to scattering)

$\pi\pi \to \pi\pi$



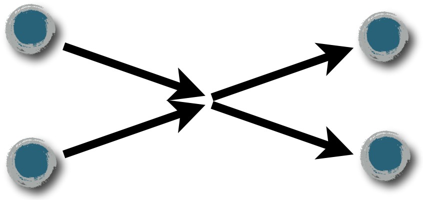


same pole,
different couplings

Optical theorem

- ☐ One can write the angular-momentum-projected amplitude as

$$\begin{aligned} i\mathcal{M}_\ell(s) &= \langle \mathrm{out}|\mathrm{in}\rangle_\ell - \mathrm{disc.} \\ &= \langle \mathrm{out}|\mathrm{in}\rangle_\ell - \langle \mathrm{out}|\mathrm{out}\rangle_\ell \end{aligned}$$



$$s = E_{cm}^2$$

- ☐ This then leads to the identities

$$2\,\mathrm{Im}[\mathcal{M}_\ell(s)] = \langle \mathrm{out}|\mathrm{out}\rangle_\ell + \langle \mathrm{in}|\mathrm{in}\rangle_\ell - \langle \mathrm{out}|\mathrm{in}\rangle_\ell - \langle \mathrm{in}|\mathrm{out}\rangle_\ell$$

$$|\mathcal{M}_\ell(s)|^2 = (\langle \mathrm{in}|\mathrm{out}\rangle_\ell - \langle \mathrm{out}|\mathrm{out}\rangle_\ell)(\langle \mathrm{out}|\mathrm{in}\rangle_\ell - \langle \mathrm{out}|\mathrm{out}\rangle_\ell)$$

- ☐ Integrating out-states in (2) and substituting $\mathbf{1} = \int |\mathrm{out}\rangle\langle \mathrm{out}|$ then yields

$$\int |\mathcal{M}_\ell(s)|^2 = 2\,\mathrm{Im}[\mathcal{M}_\ell(s)]$$

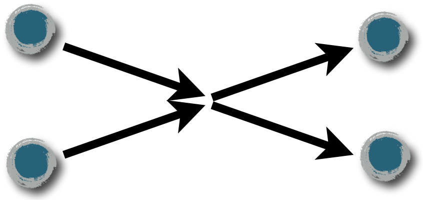
- ☐ For two-particle energies the integral becomes a simple factor…

$$\rho(s)\,|\mathcal{M}_\ell(s)|^2 = \mathrm{Im}[\mathcal{M}_\ell(s)] \qquad \rho(s) = \frac{\sqrt{1-4m^2/s}}{32\pi}$$

Analyticity

- Instead of $|\mathcal{M}(s)|^2$ → analytically continue the ***amplitude*** itself

  For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?



- The optical theorem tells us…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \mathrm{Im}\,\mathcal{M}_\ell(s)$$

where $\rho(s) = \dfrac{\sqrt{1-4m^2/s}}{32\pi}$ is the two-particle phase space

- Unique solution is…

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

K matrix (short distance)

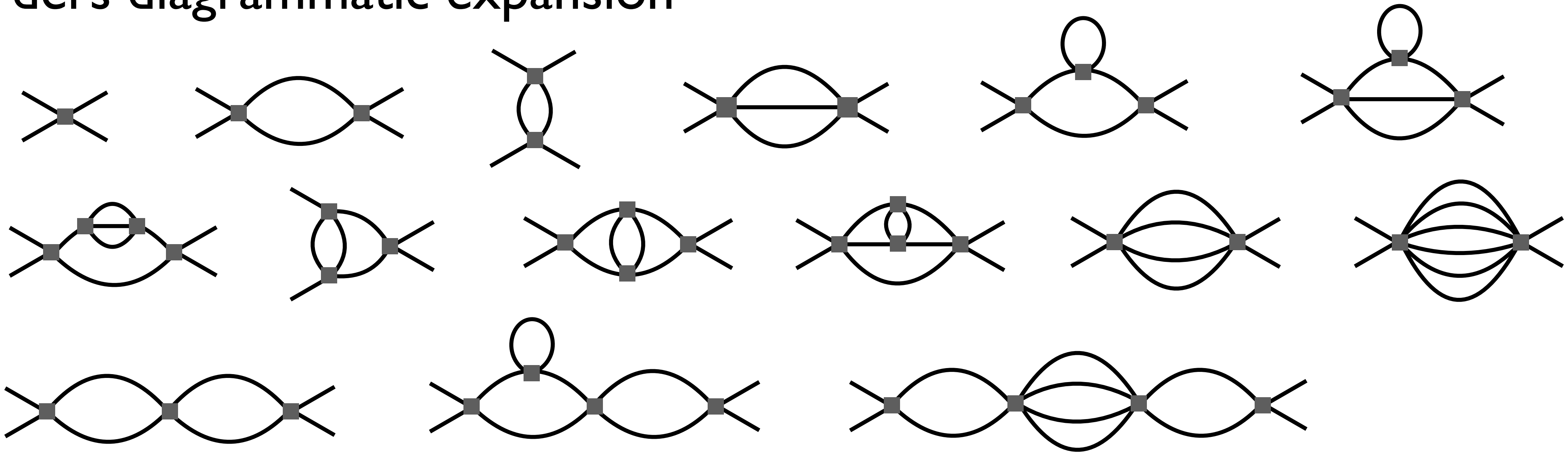
phase-space cut (long distance)

Key message: *The scattering amplitude has a square-root branch cut*

Bethe Salpeter equation

- All orders diagrammatic expansion

$\mathcal{M} =$



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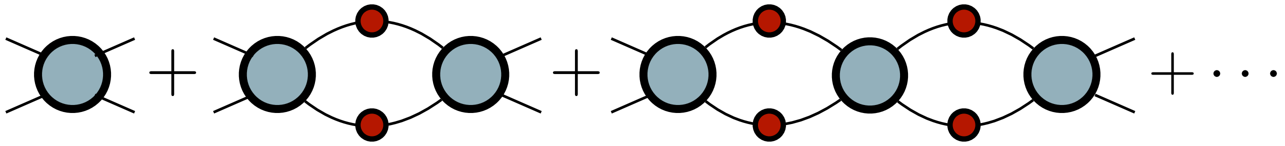

- Construct ○ and ─●─ such that this is true

○ $=$

─●─ $=$

Bethe Salpeter equation

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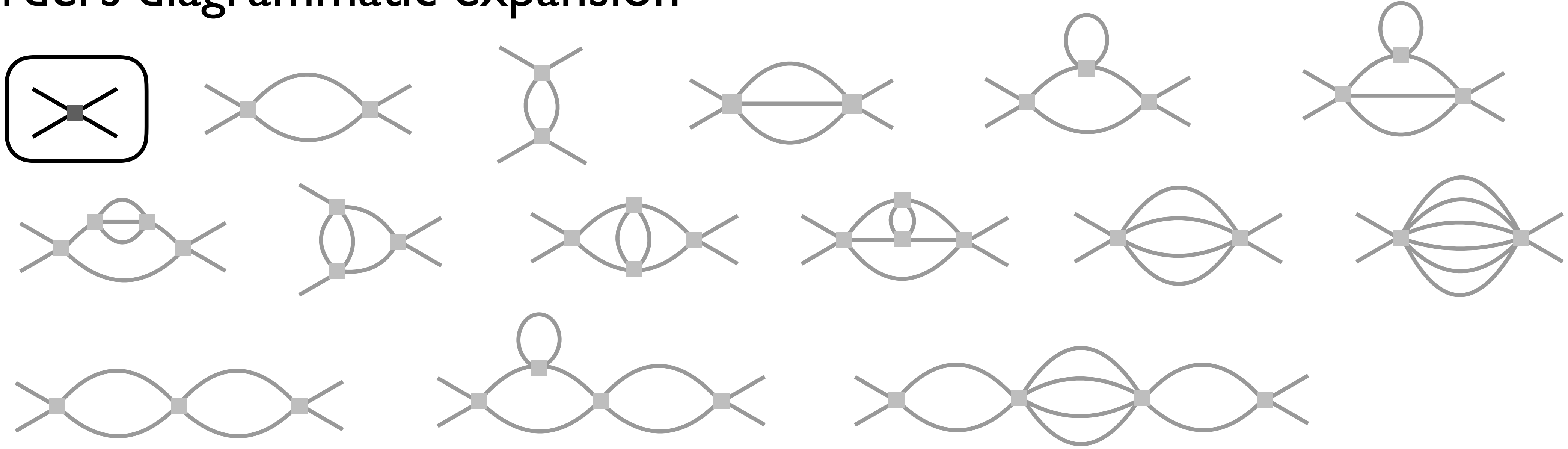

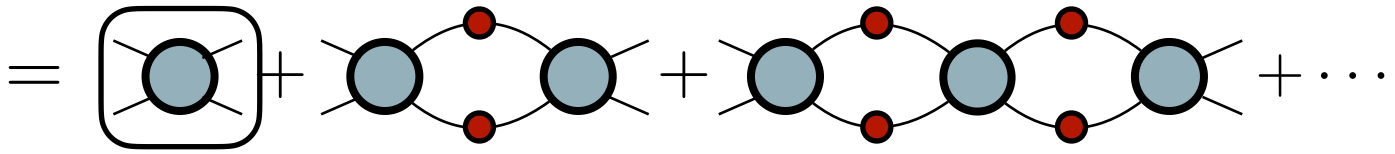

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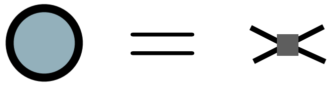

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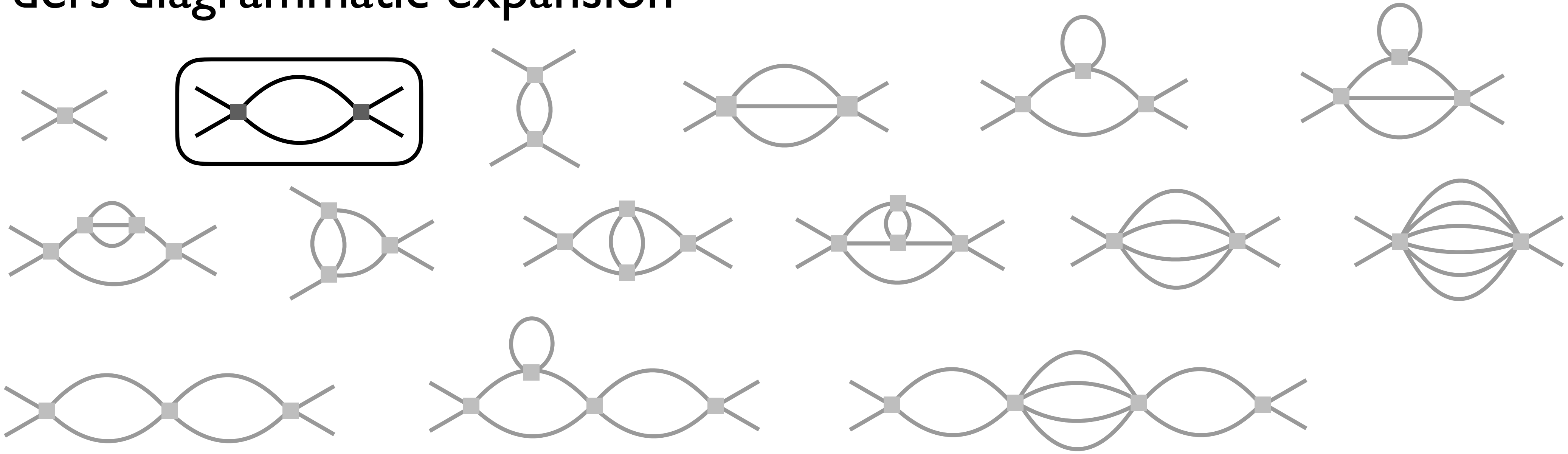

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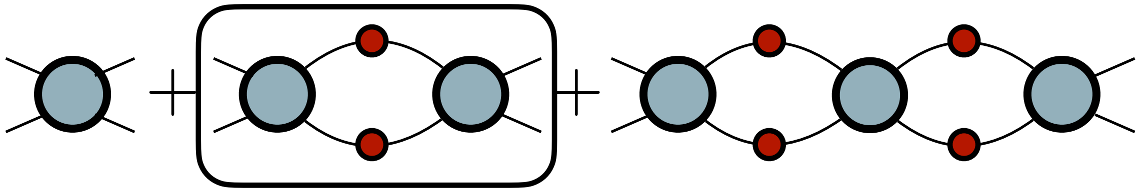

$+ \cdots$

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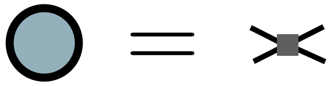

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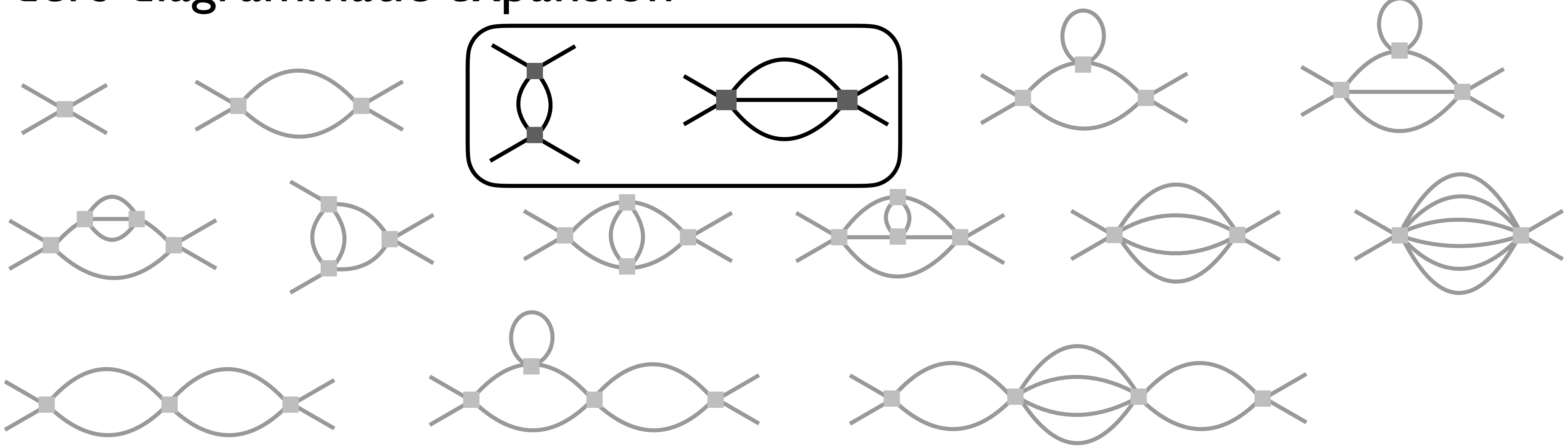

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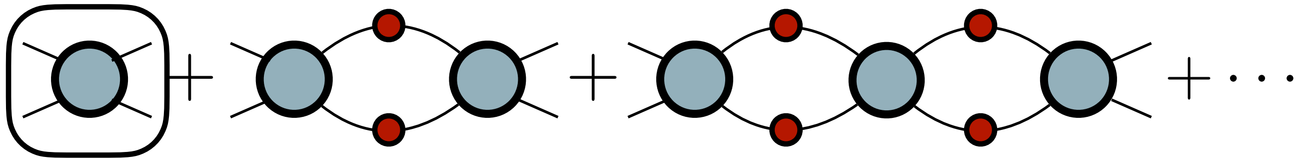

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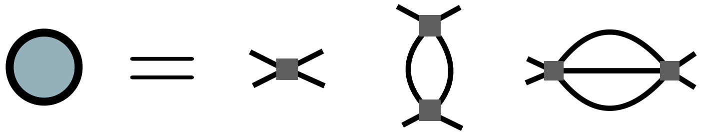

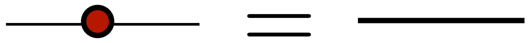

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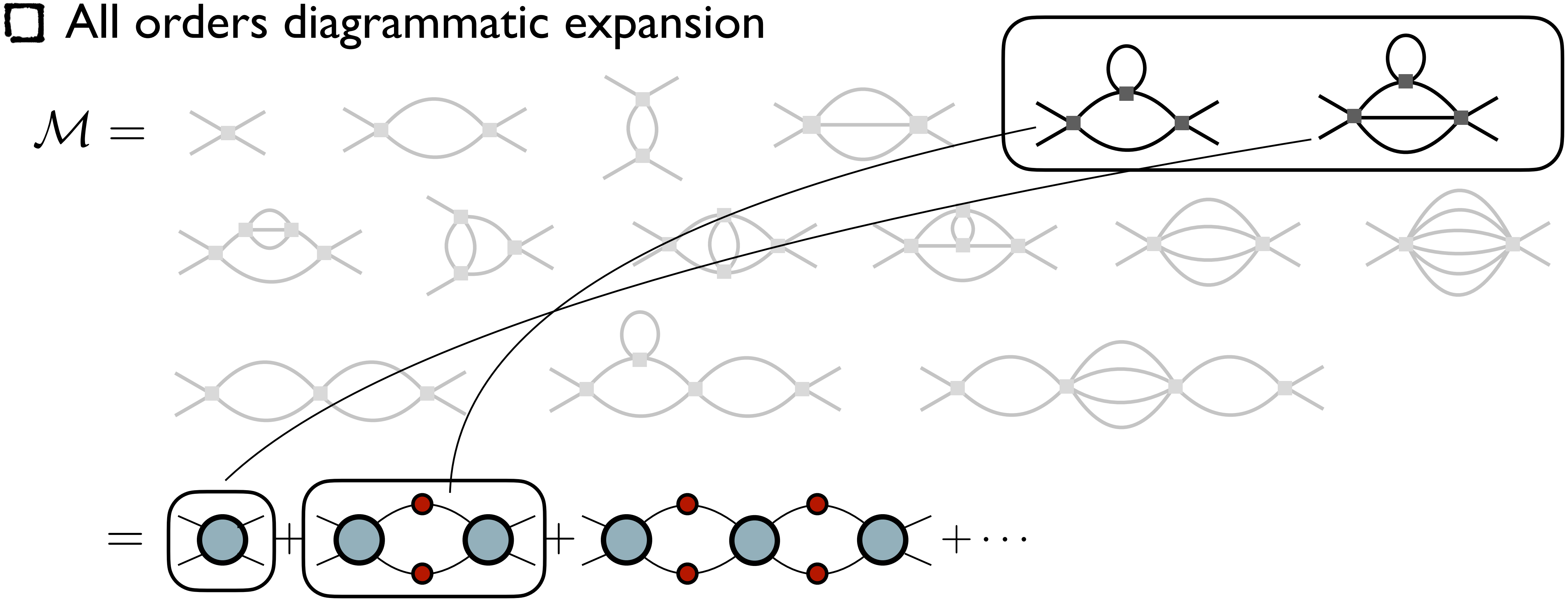

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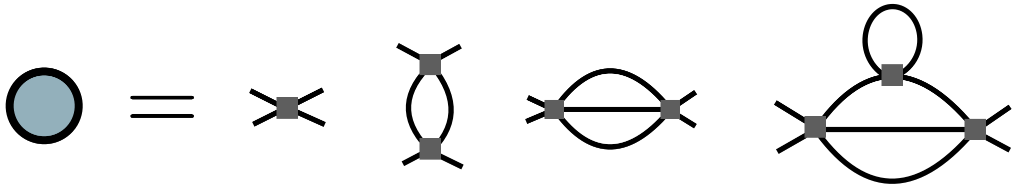

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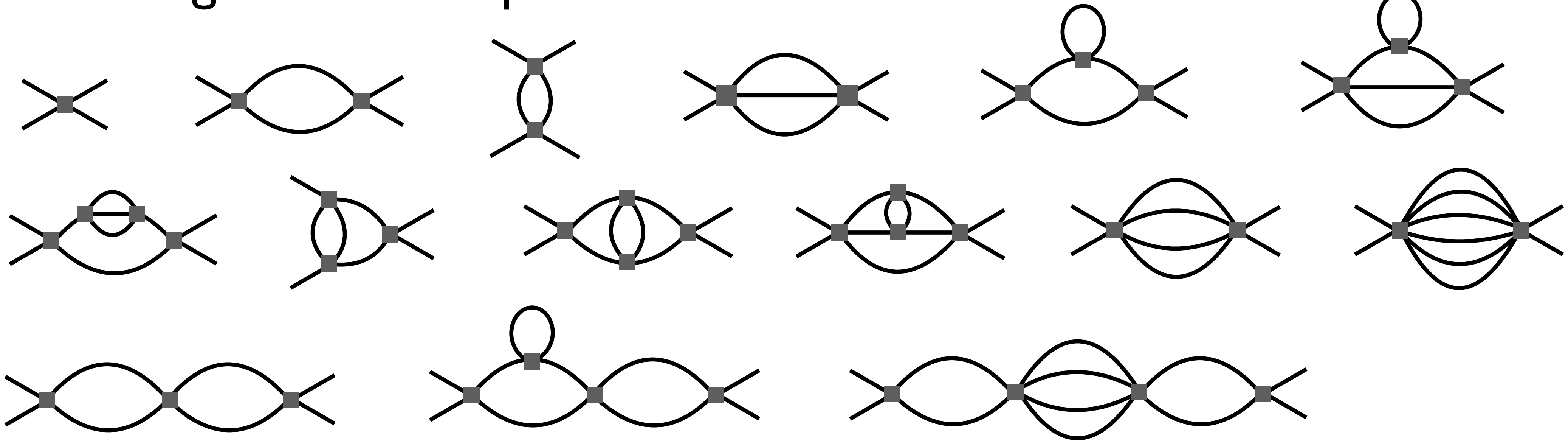

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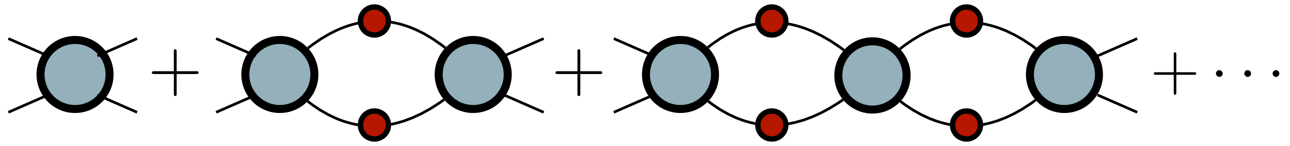

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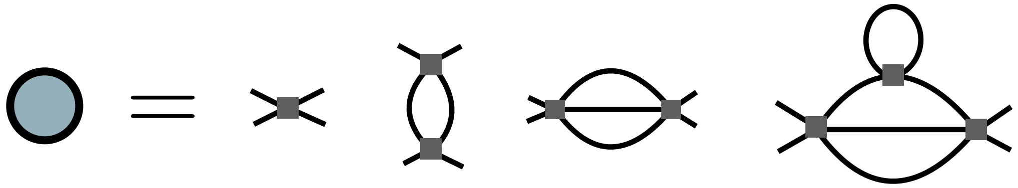
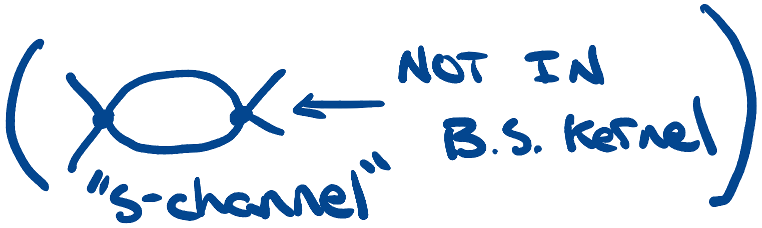
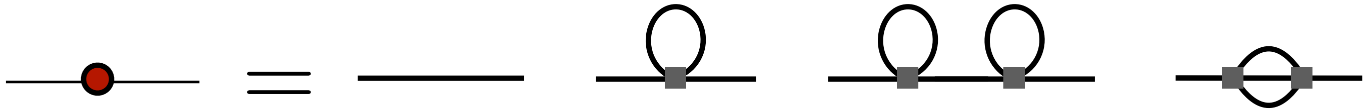




NOT IN B.S. kernel

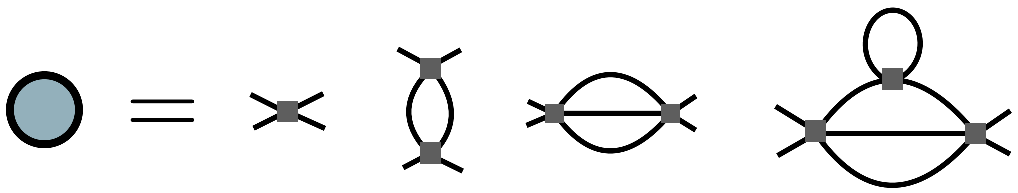
Bethe Salpeter kernel

(2PI in the s-channel)



Fully dressed propagator

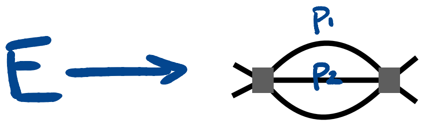
Bethe Salpeter kernel

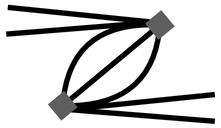


$= B(s)$

- ☐ Analyticity of B.S. kernel

time-ordered (old fashioned) perturbation theory





$$\int_{\boldsymbol{p}_1,\boldsymbol{p}_2} \frac{1}{E-\omega_{\boldsymbol{p}_1}-\omega_{\boldsymbol{p}_2}-\omega_{\boldsymbol{P}-\boldsymbol{p}_1-\boldsymbol{p}_2}}$$

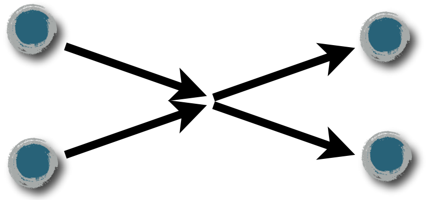
$$\int_{\boldsymbol{p}_1,\boldsymbol{p}_2} \frac{1}{-E-\omega_{\boldsymbol{p}_1}-\omega_{\boldsymbol{p}_2}-\omega_{\boldsymbol{P}-\boldsymbol{p}_1-\boldsymbol{p}_2}}$$

$\omega_{p_1} = \sqrt{\vec{p}_1^{\,2} + M^2}$

- ☐ Real and analytic for $s < (3m)^2$

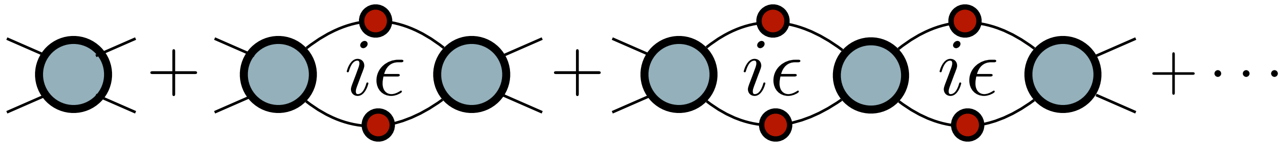
no $i\epsilon$ needed if pole is not satisfied

Analyticity of $\mathscr{M}$ (from B.S. kernel)



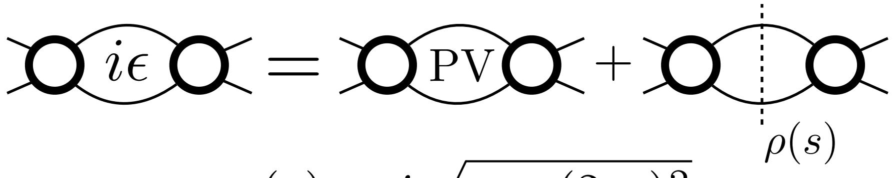
For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?

$\mathcal{M}(s) \equiv$



non-analytic:
on-shell particles = singularities

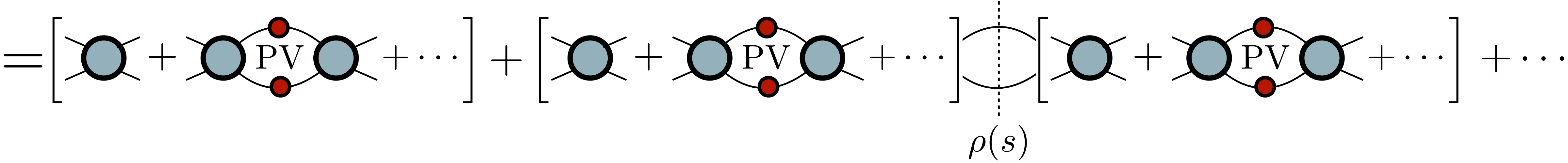
— propagating pion



$\rho(s) \propto i\sqrt{s-(2m)^2}$

cutting rule

defines the *K* ***matrix***



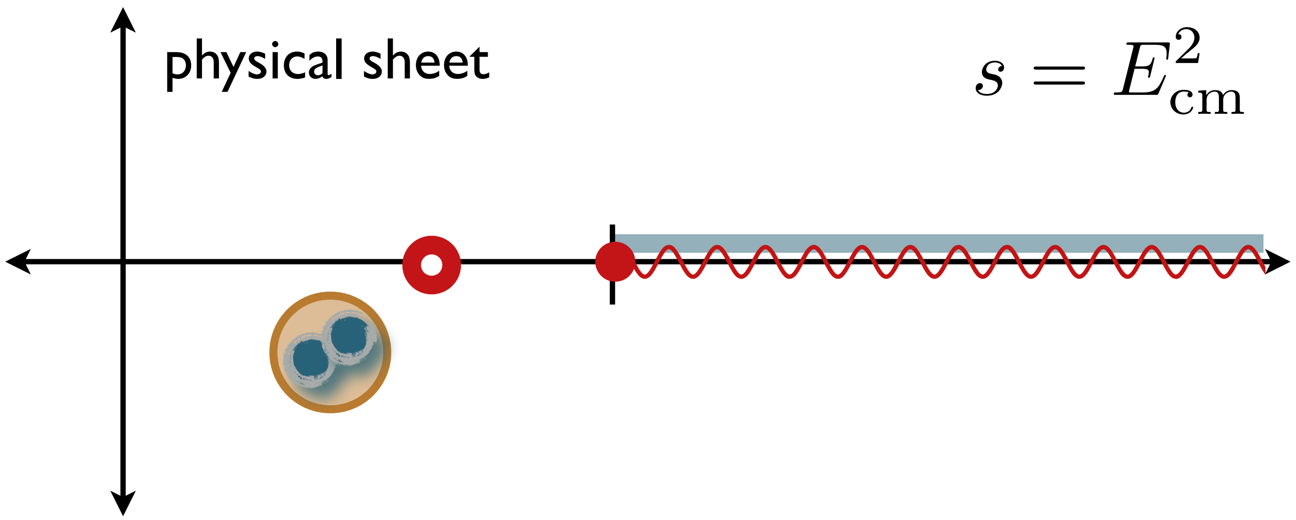
$$= \mathcal{K}(s) + \mathcal{K}(s)\rho(s)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - \rho(s)}$$

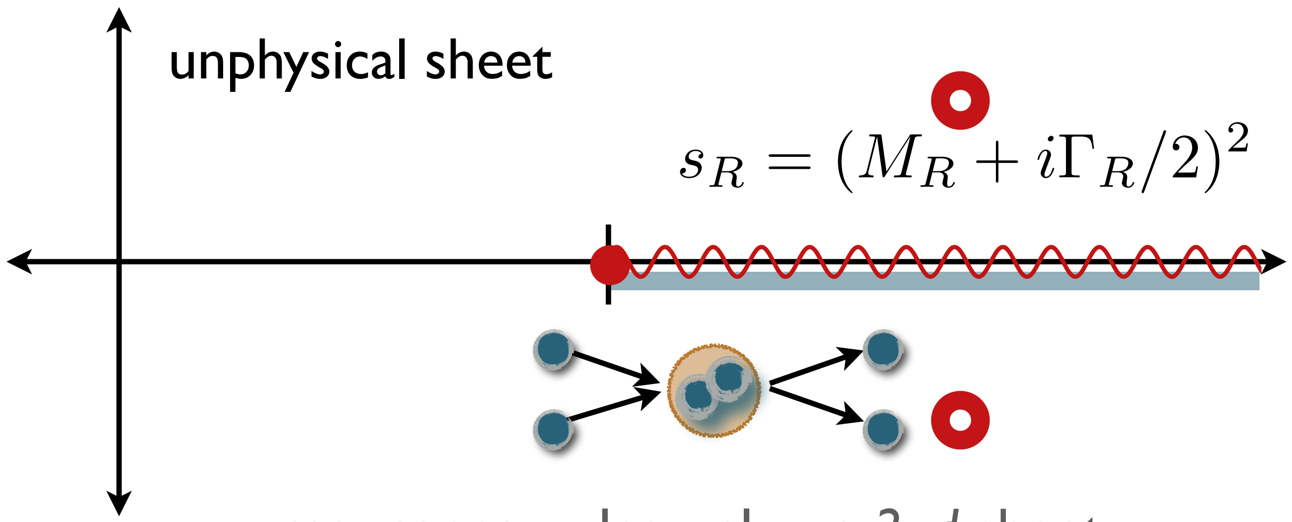
branch-cut singularity
$\sqrt{s-(2m)^2}$

Cuts and sheets

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - \rho(s)} \propto \frac{1}{p \cot \delta_\ell(s) - ip} \propto e^{2i\delta_\ell(s)} - 1 \qquad \rho(s) \propto i\sqrt{s-(2m)^2}$$

☐ Each channel generates a *square-root cut* → doubles the number of sheets





resonance poles only on *2nd* sheet

☐ Important lessons:

Details of analyticity = important for quantitative understanding

Possible to separate…

(i) long-distance kinematic singularities

(ii) short-distance/microscopic physics (depending on interaction details)

Five-day outline

- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} volume effects
  - ☑ Mass, matrix element in $\lambda\phi^4, g\phi^3$
  - ☑ Pion mass
  - ☑ LO-HVP for $(g-2)_\mu$
  - ☐ Bethe-Salpeter kernel
- ☐ $2 \to 2$ finite-volume (f.v.) formalism
  - ☑ Scattering basics
  - ☐ Derivation
  - ☐ Generalizations
  - ☐ Applications
- ☐ $(1+)\mathcal{J} \to 2$ f.v. formalism
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  - ☐ Derivation
  - ☐ Testing the result
  - ☐ Numerical explorations
- ☐ Non-local matrix elements
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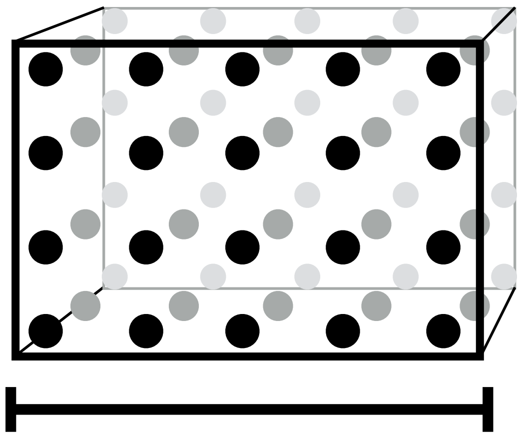
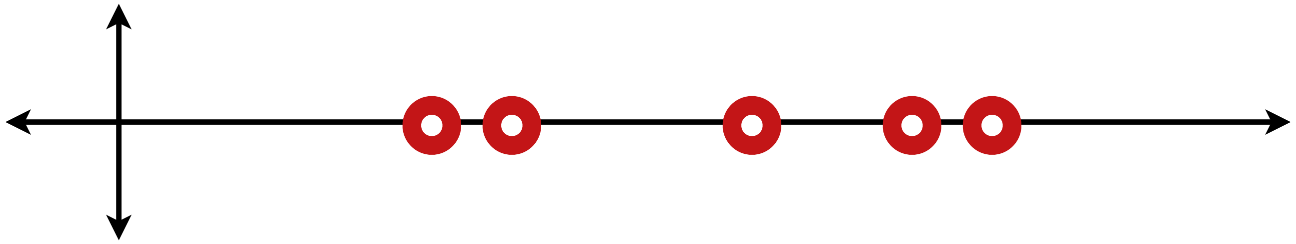
  - ☐ Applications
- ☐ $3 \to 3$ f.v. formalism
  - ☐ New complications
  - ☐ Derivation ($E_n(L)$ to $\mathcal{K}_{\text{df},3}$)
  - ☐ Integral equations ($\mathcal{K}_{\text{df},3}$ to $\mathcal{M}_3$)
  - ☐ Testing the result
  - ☐ Numerical explorations/calculations
- ☐ Finite-volume QED + QCD
  - ☐ Different formalisms
  - ☐ Basic examples



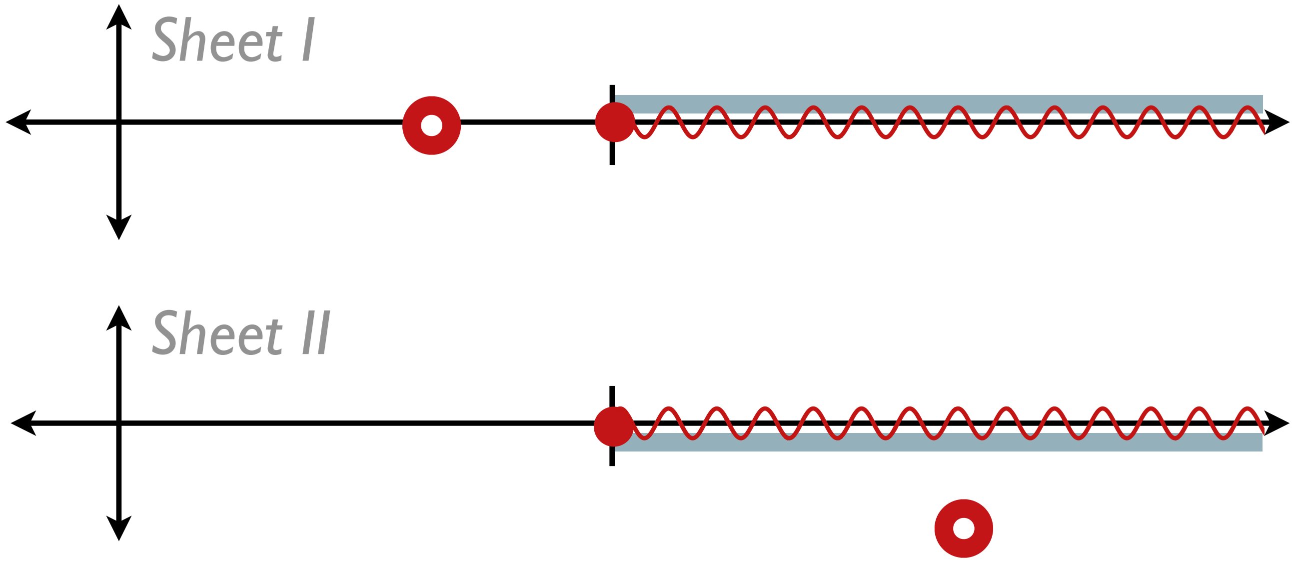
☐ The ***finite volume***…

- ○ *Discretizes* the spectrum
- ○ *Eliminates* the branch cuts and extra sheets
- ○ *Hides* the resonance poles

Finite-volume analytic structure



Infinite-volume analytic structure



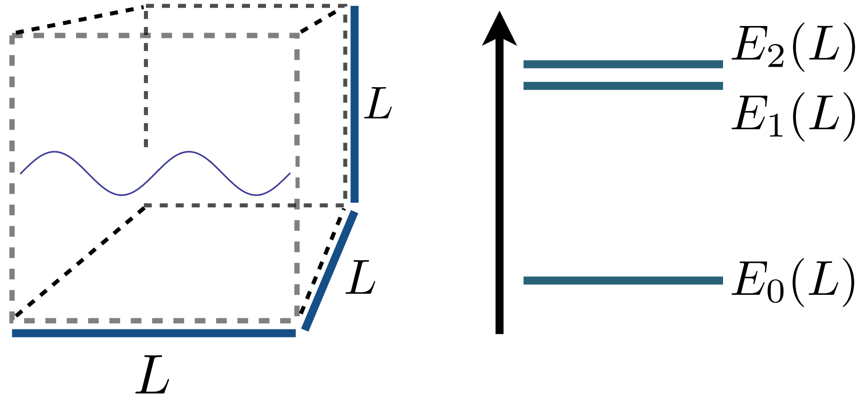
☐ LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z^*_{n,i}$$

☐ Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***

The finite-volume as a tool

☐ Finite-volume set-up



☐ **cubic**, spatial volume (extent *L*)

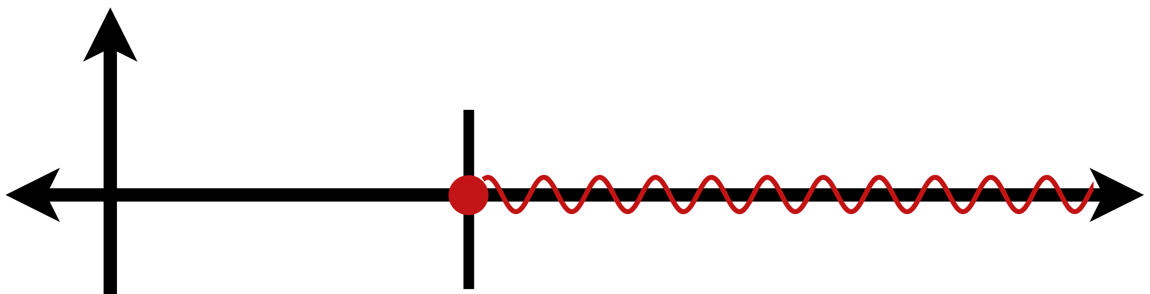
☐ **periodic**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

☐ *L* is large enough to neglect $\underline{e^{-M_\pi L}}$

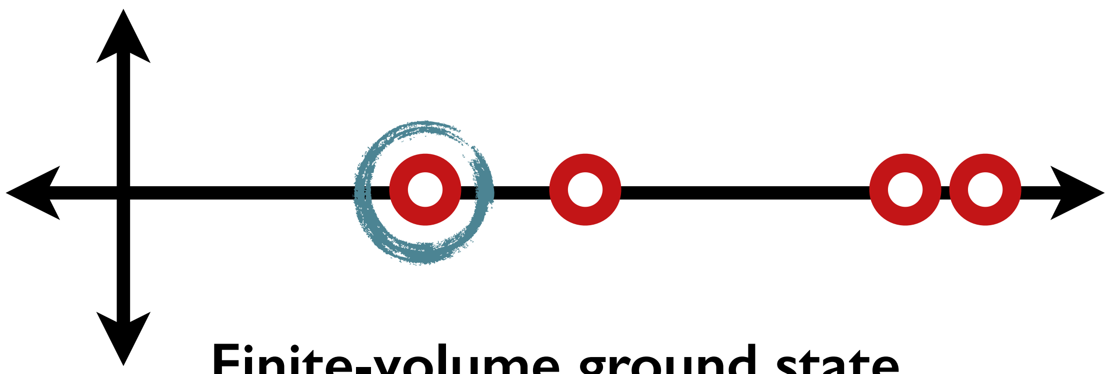
☐ *T* and lattice also negligible

☐ Scattering leaves an ***imprint*** on finite-volume quantities



Infinite-volume threshold

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$



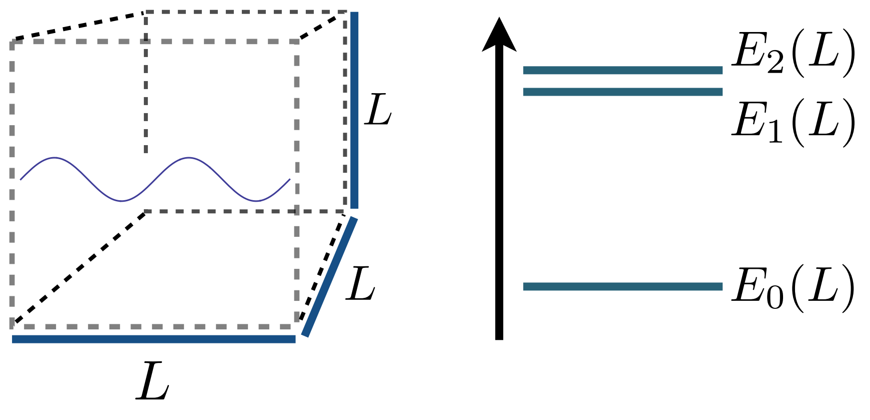
Finite-volume ground state

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

The finite-volume as a tool

☐ Finite-volume set-up



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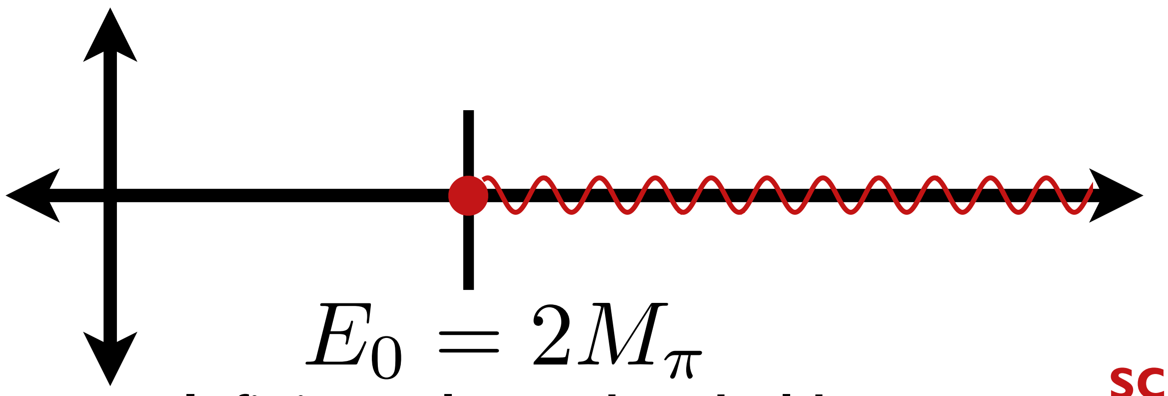
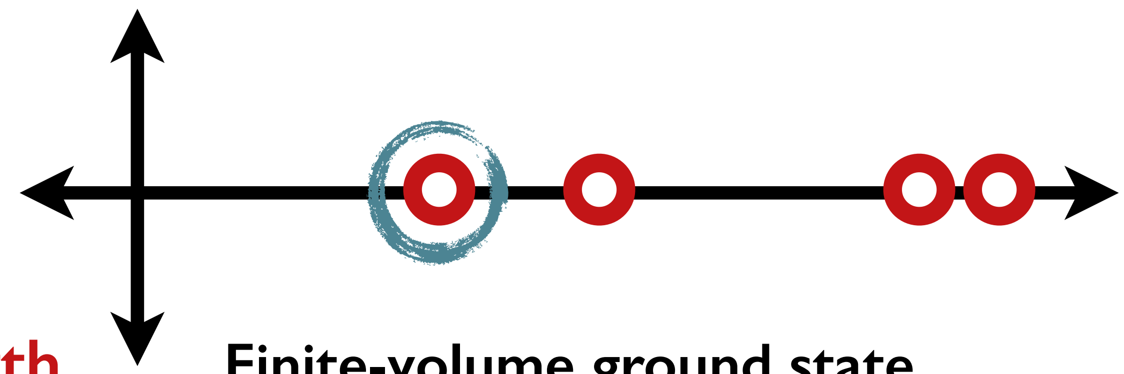


$E_0 = 2M_\pi$

Infinite-volume threshold



Finite-volume ground state

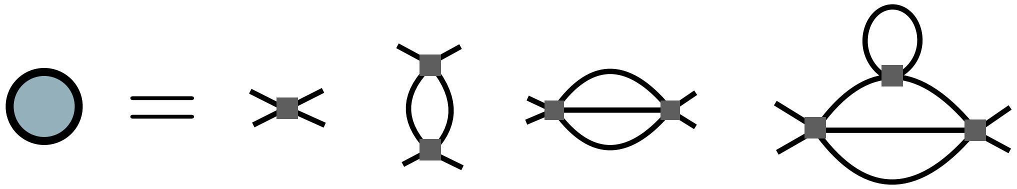
scattering length

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

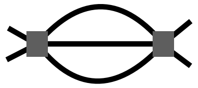
Bethe-Salpeter kernel



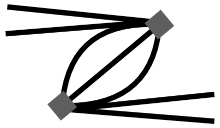
$$= B(s)$$

- ☐ Exponentially suppressed volume effects e^{-mL} for $(2m)^2 < s < (3m)^2$

  time-ordered (old fashioned) perturbation theory



$$\frac{1}{L^6}\sum_{\boldsymbol{p}_1,\boldsymbol{p}_2}\frac{1}{E-\omega_{\boldsymbol{p}_1}-\omega_{\boldsymbol{p}_2}-\omega_{\boldsymbol{P}-\boldsymbol{p}_1-\boldsymbol{p}_2}}$$



$$\frac{1}{L^6}\sum_{\boldsymbol{p}_1,\boldsymbol{p}_2}\frac{1}{-E-\omega_{\boldsymbol{p}_1}-\omega_{\boldsymbol{p}_2}-\omega_{\boldsymbol{P}-\boldsymbol{p}_1-\boldsymbol{p}_2}}$$

- ☐ Recall…

$$\int_k [\text{smooth}]\; e^{iLk\cdot n} = \mathcal{O}(e^{-mL})$$

Derivation

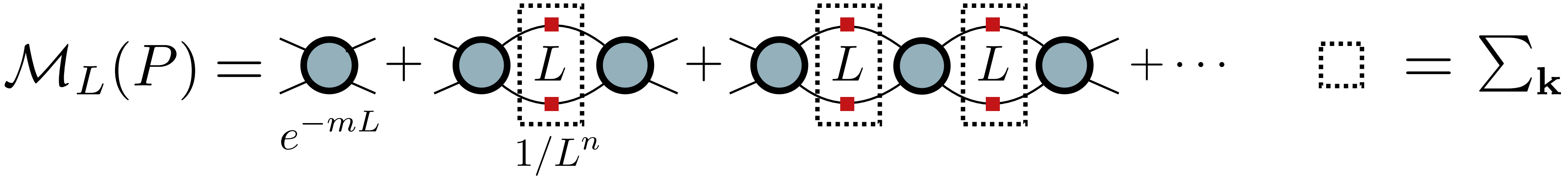
☐ Consider the finite-volume correlator:

$\mathcal{M}(s)$

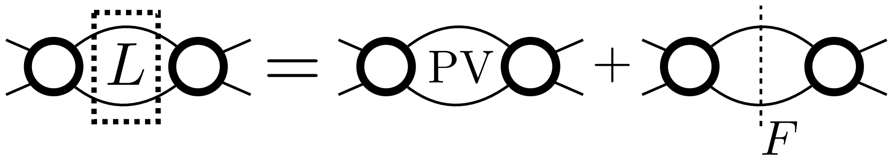
probability amplitude

$\mathcal{M}_L(P)$

poles give f.v. spectrum



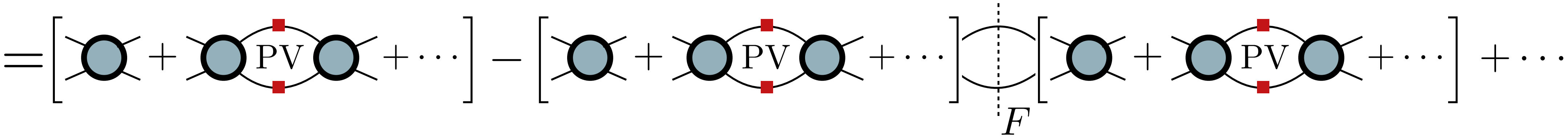
For two-particle energies $(2m)^2 < s < (4m)^2$, what is the *L* dependence?



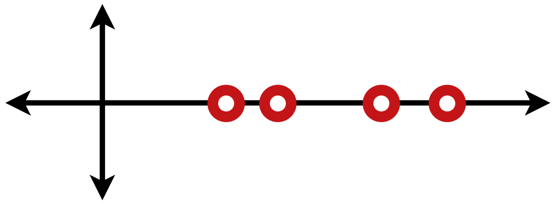
Cut projects loop to ***on-shell energies***

F = matrix of known geometric functions

Defines the K matrix



$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$

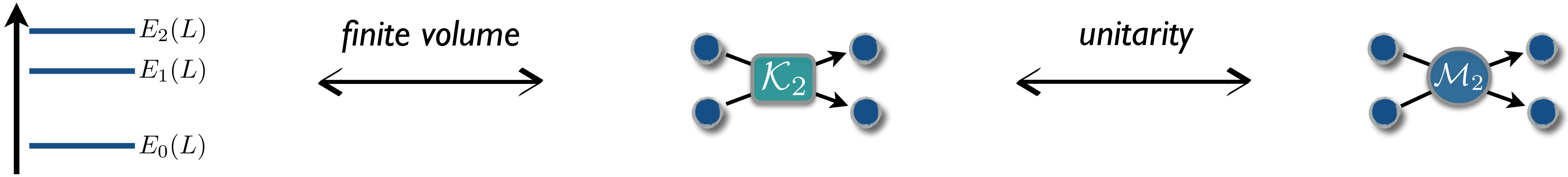


- Lüscher (1986)
- Kim, Sachrajda, Sharpe (2005)
- MTH, Sharpe (*coupled channels*, 2012)

Result

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

$F(P,L) \equiv$ Matrix of known geometric functions



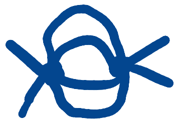
Holds only for two-particle energies $s < (4m)^2$

Neglects e^{-mL}

Generalized to *non-degenerate masses, multiple channels, spinning particles*

Encodes angular momentum mixing

For $s>(4m)^2$ we have to worry about ...

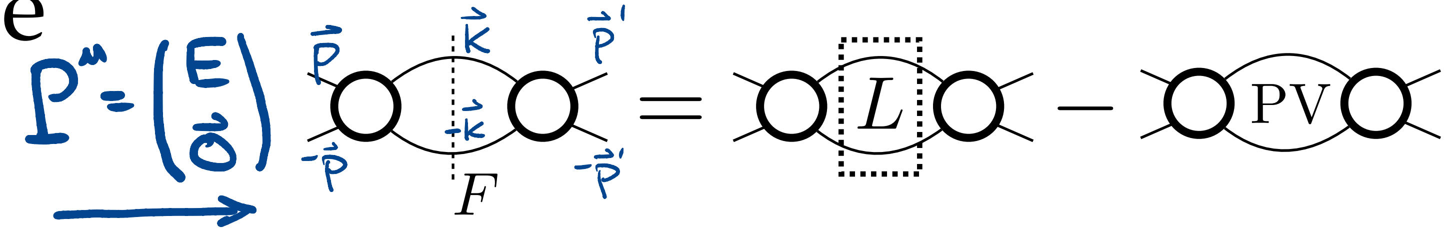


$= \sum \frac{1}{E-\omega-\omega-\omega-\omega}$

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
Li, Liu (2013) • Briceño (2014)

Cutting rule

$$P^\mu = \begin{pmatrix} E \\ \vec{0} \end{pmatrix}$$



$$B \otimes F \otimes B = \frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right] B(s,\boldsymbol{p},\boldsymbol{k})D(k)D(P-k)B(s,\boldsymbol{k},\boldsymbol{p}')$$

$$= \frac{1}{2}\Delta_{\vec{k}} \frac{B(s,\sqrt{s/4-m^2}\,\hat{p},\sqrt{s/4-m^2}\,\hat{k})\cdot B(\ldots)}{(2\omega_{\vec{k}})^2(E-2\omega_{\vec{k}})} + \Delta_{\vec{k}}\ \text{smooth}$$

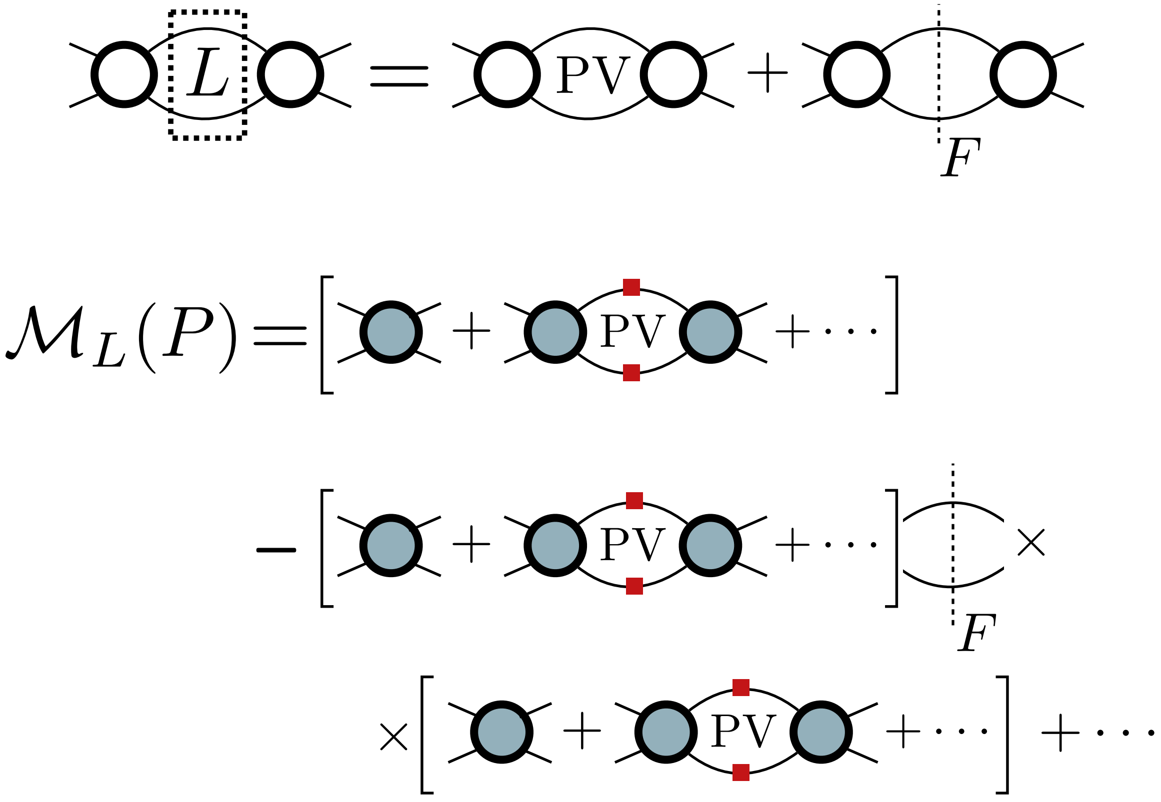
$$s = E_{cm}^2$$

$$E_{cm} = 2\sqrt{p_{\text{on-shell}}^2 + m^2} \Longrightarrow p_{\text{on-shell}} = \sqrt{s/4 - m^2}$$

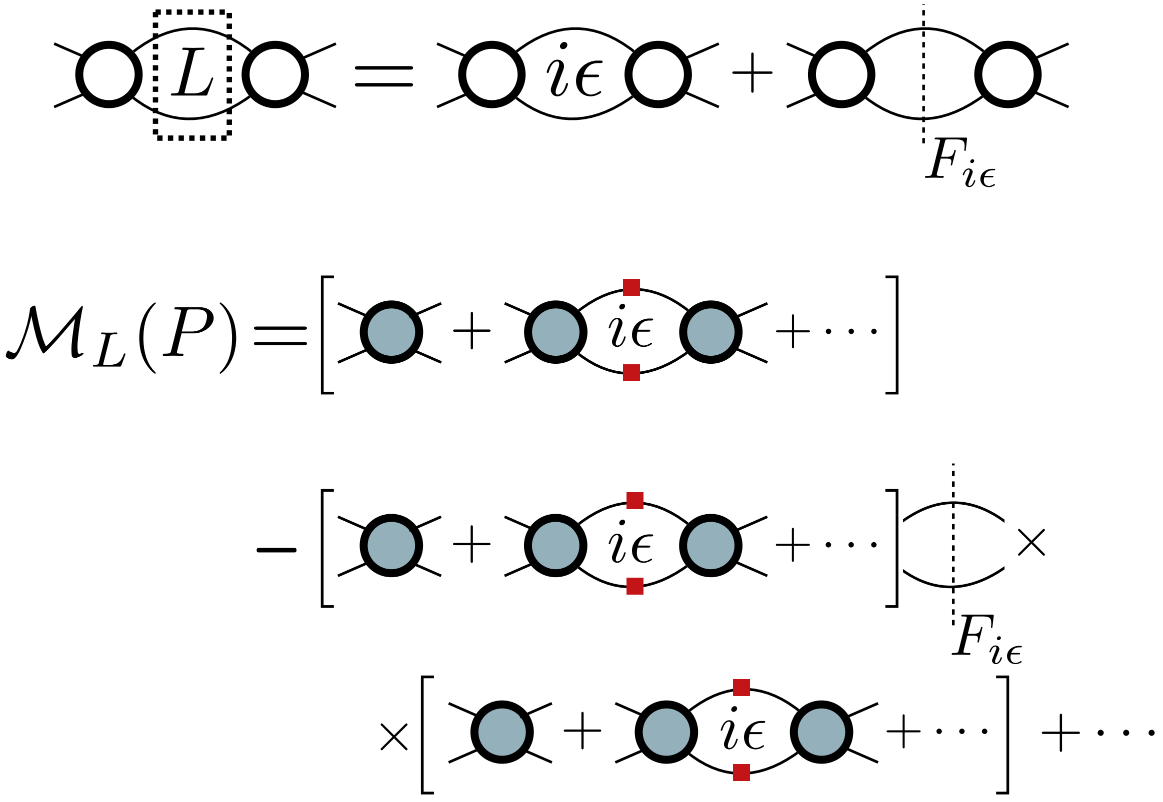
$$= Y_{\ell m}(\hat{p})\, B_{\ell m,\ell' m'}(s)\, F_{\ell' m',\ell'' m''}(E,L)\, B_{\ell'' m'',\ell''' m'''}(s)\, Y_{\ell''' m'''}(\hat{p}')$$

$$F_{\ell' m',\ell'' m''} = \frac{1}{2}\Delta_k \frac{Y_{\ell' m'}\, Y_{\ell'' m''}}{(2\omega_k)^2(E-2\omega_k)}$$

Pole prescription freedom



$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$



$$= \frac{1}{\mathcal{M}(s)^{-1} + F_{i\epsilon}(P,L)}$$

$$\mathcal{K}(s)^{-1} + F(P,L) = [\mathcal{K}(s)^{-1} - i\rho(s)] + [F(P,L) + i\rho(s)] = \mathcal{M}(s)^{-1} + F_{i\epsilon}(P,L)$$

$1/L$ expansions

$$\overset{K^{-1}}{p\cot\delta(p)} = \frac{1}{\pi L}\left[\sum_{\boldsymbol{n}} \overset{-F}{-\text{p.v.}\int d^3\boldsymbol{n}}\right]\frac{1}{\boldsymbol{n}^2 - q^2}$$

$q = \frac{pL}{2\pi}$

$E = 2\sqrt{\vec{p}^{\,2} + M^2}$

$\det[K^{-1} + F] = 0$

$S = e^{2i\delta} \Rightarrow \mathcal{M} \propto e^{2i\delta} - 1 \propto \frac{1}{p\cot\delta - ip} \Rightarrow K^{-1} \propto p\cot\delta$

$$p\cot\delta = \underbrace{-1/a}_{\text{scattering length}} + \mathcal{O}(p^2) \overset{!}{=} \frac{1}{\pi L}\left[\Sigma - \int\right]\cdots = -\frac{1}{\pi L}\frac{1}{q^2} + \mathcal{O}(q^0)$$

$$\Rightarrow p^2 = \frac{4\pi a}{L^3} \Longrightarrow \underline{E_0(L) = 2M + \frac{4\pi a}{ML^3} + \mathcal{O}(1/L^4)}$$

Huang + Yang

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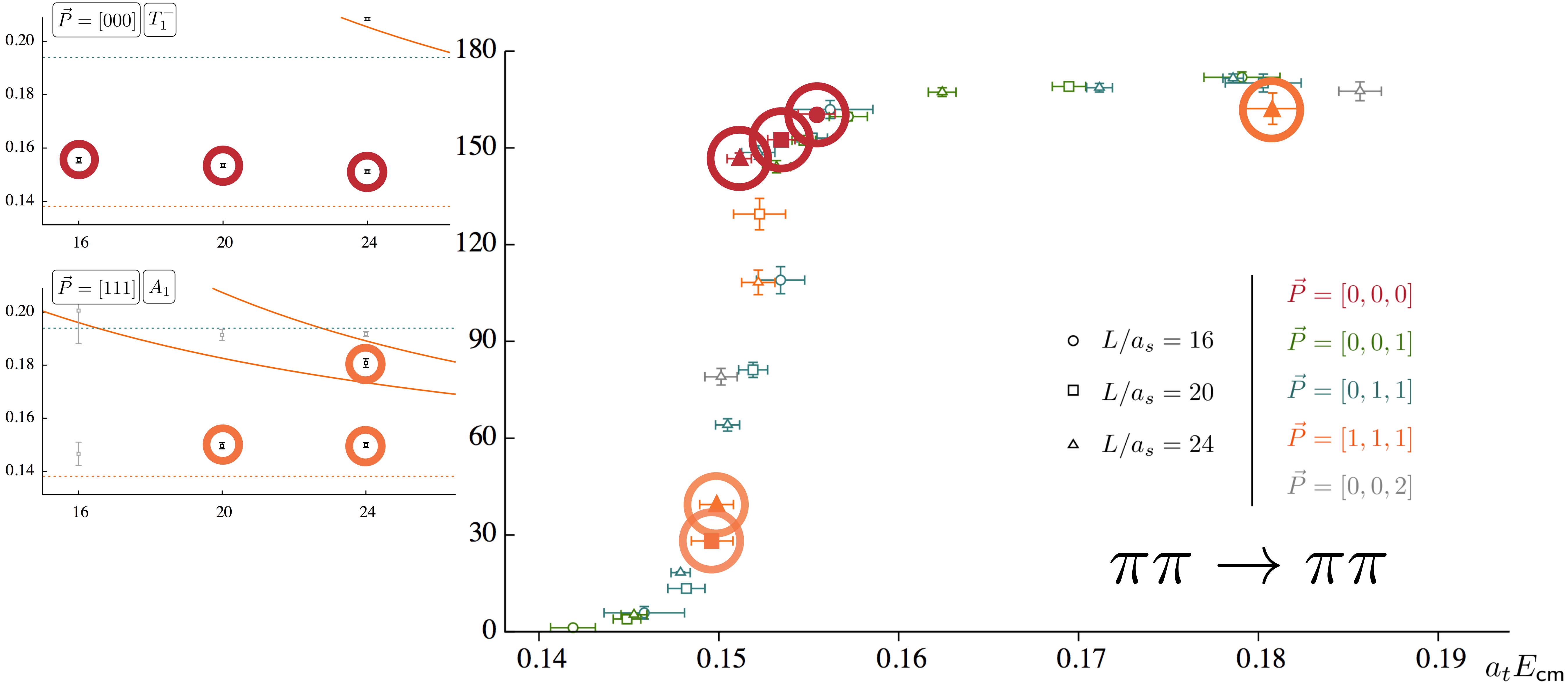
  - ☐ Basic examples

Using the result

☐ Single-channel case *(pions in a p-wave)*

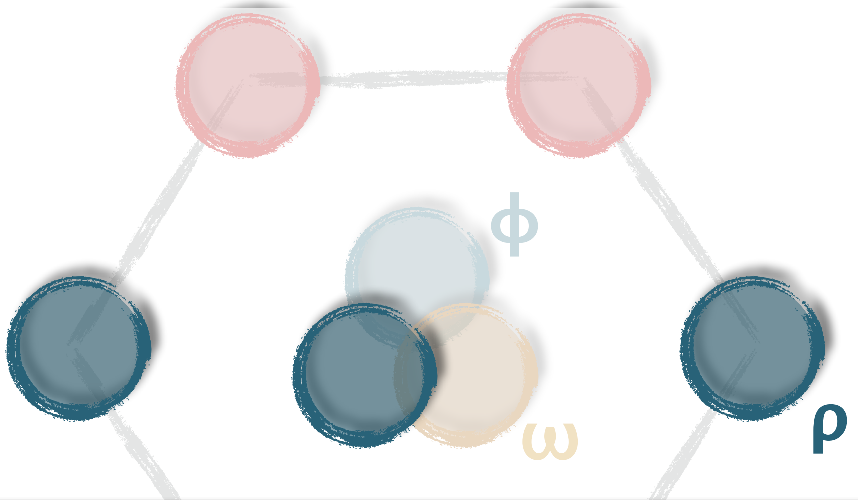
$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$

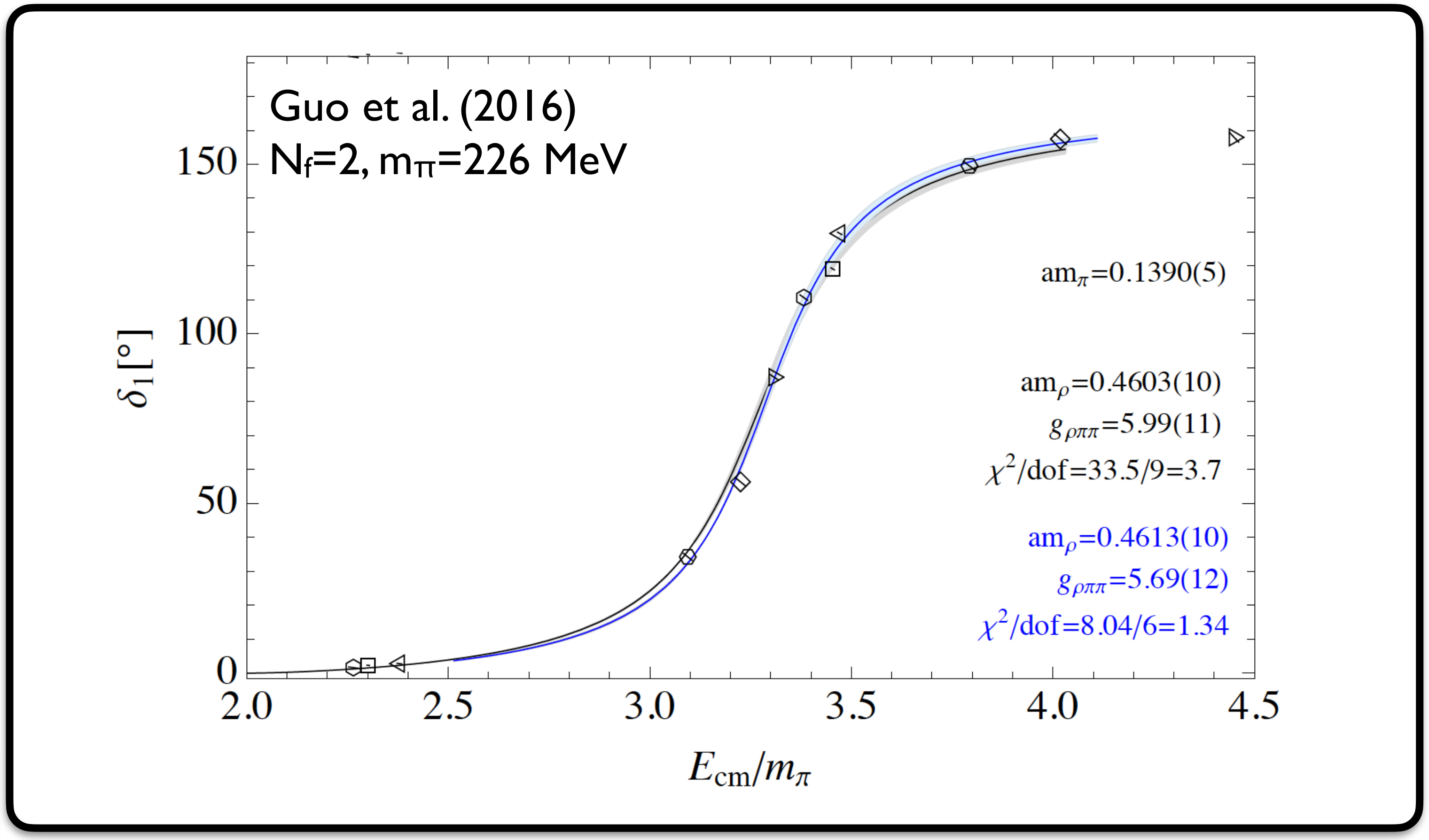


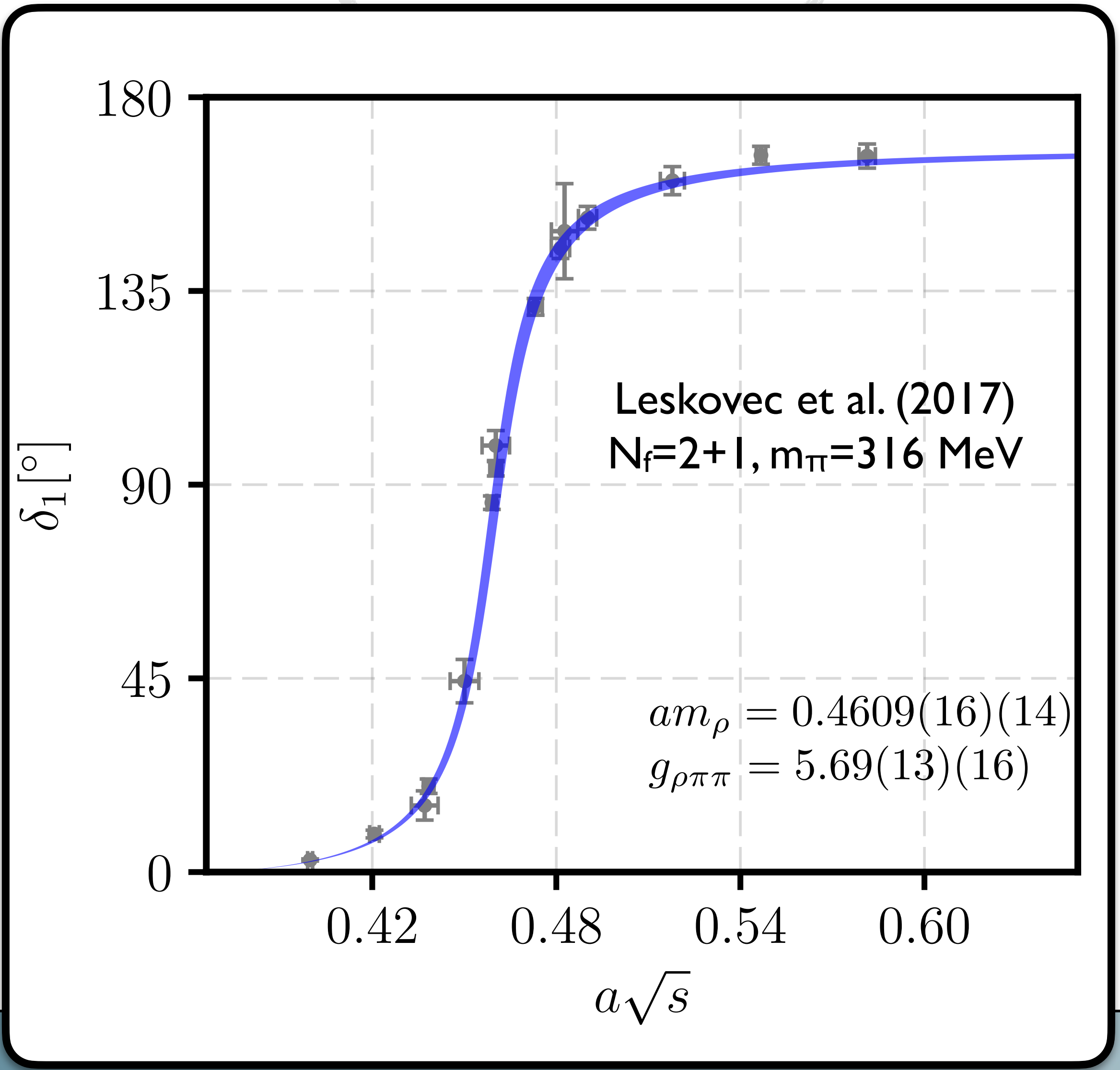
• Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505 •

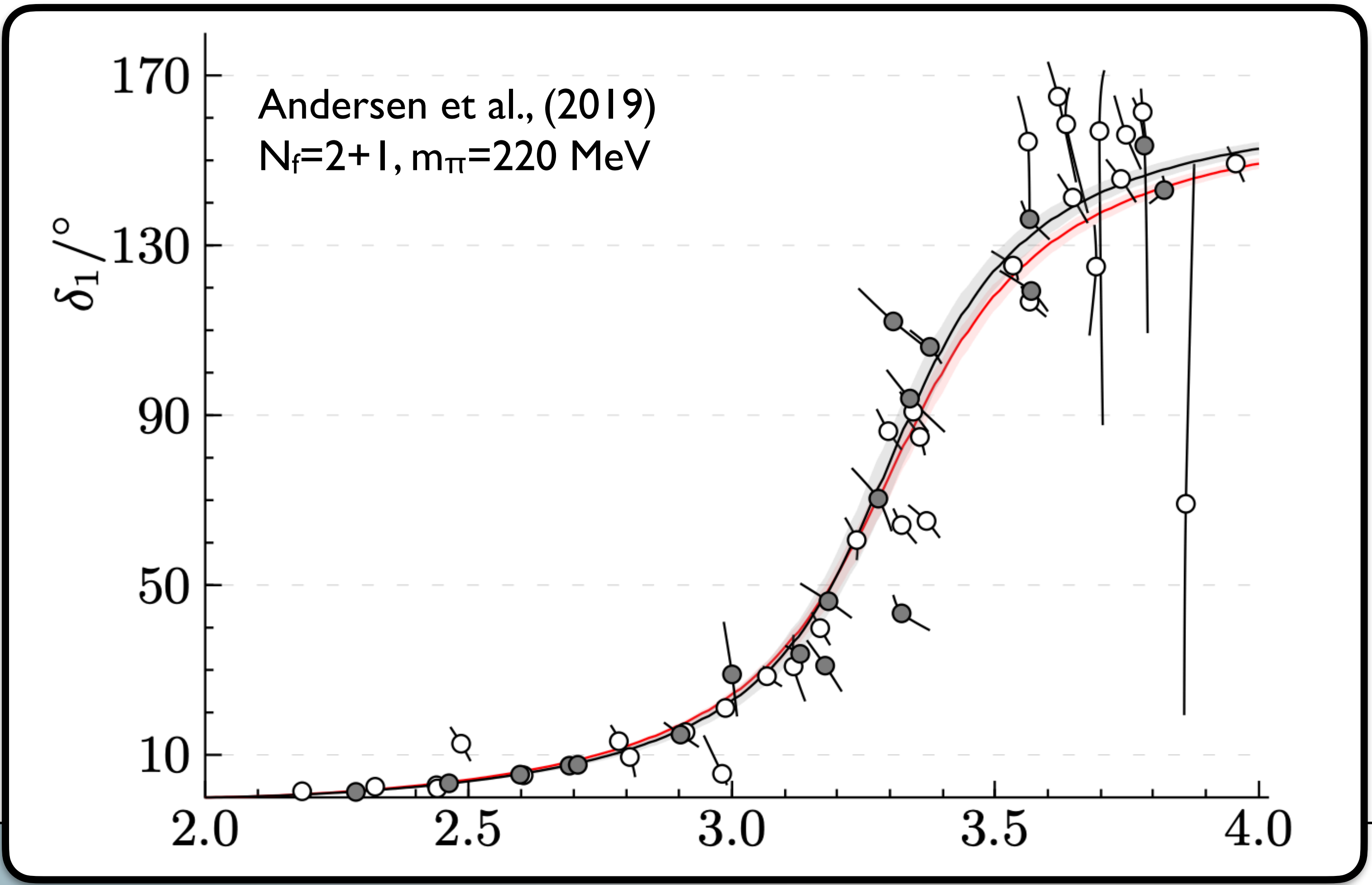
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$









$\rho \to \pi\pi$

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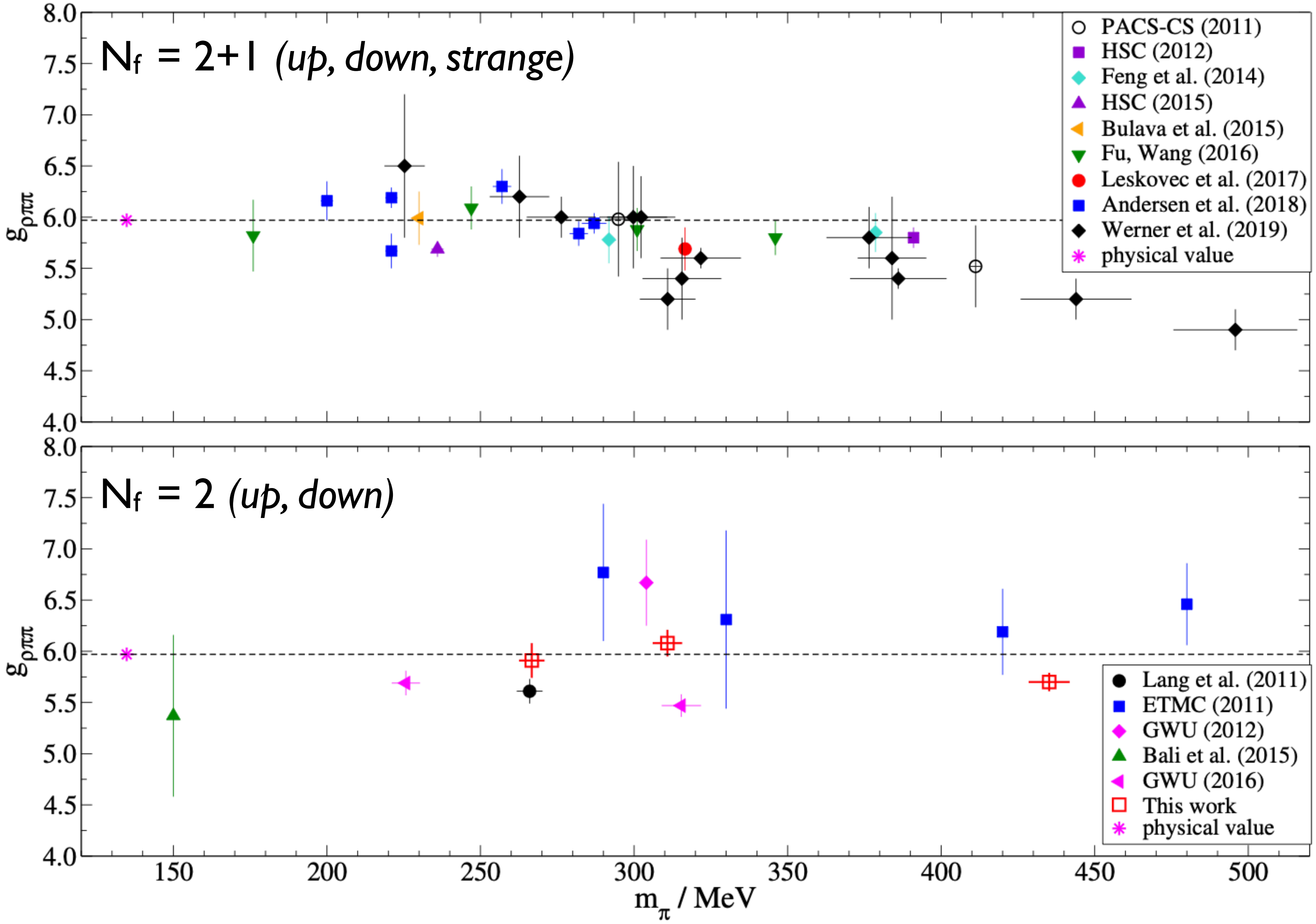
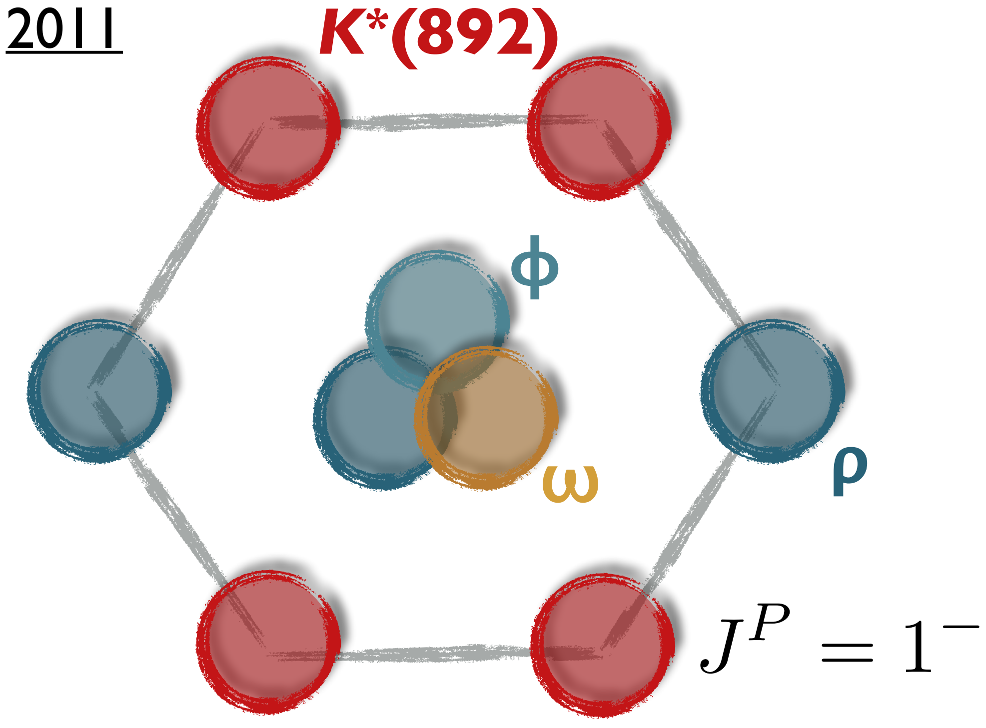


- Erben et al., Phys.Rev. **D** 101 (2020) 5, 054504 •

$\rho \to \pi\pi$

- ☐ CP-PACS/PACS-CS 2007, 2011
- ☐ ETMC 2010
- ☐ Lang et al. 2011
- ☐ HadSpec 2012, **2016**
- ☐ Pellisier 2012
- ☐ RQCD 2015
- ☐ Guo et al. 2016
- ☐ Fu et al. 2016
- ☐ Bulava et al. 2016
- ☐ Alexandrou et al. 2017
- ☐ Andersen et al. 2018
- ☐ Fischer et al. 2020
- ☐ Erben et al. 2020



$\kappa \to K\pi$
$K^* \to K\pi$

- ☐ Lang et al. 2012
- ☐ Prelovsek et al. 2013
- ☐ **Wilson et al. 2015**
- ☐ RQCD 2015
- ☐ **Brett et al. 2018**
- ☐ Wilson et al. 2019
- ☐ Rendon et al. 2020

$b_1 \to \pi\omega, \pi\phi$

- ☐ **Woss et al. 2019**

$a_0(980) \to \pi\eta, K\overline{K}$

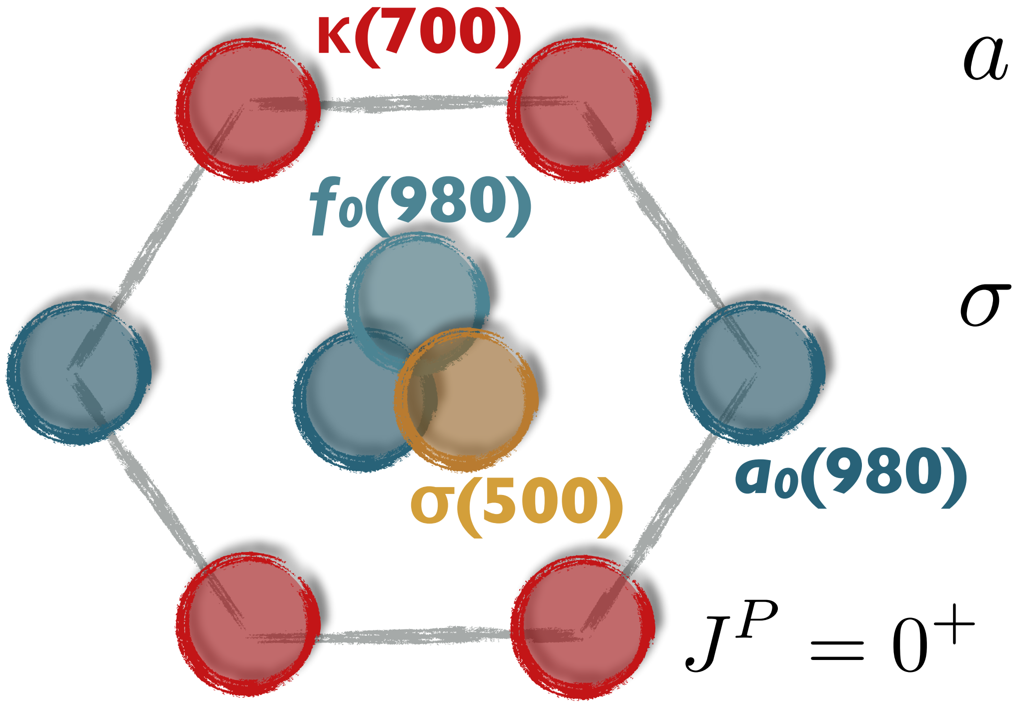
- ☐ **Dudek et al. 2016**

$\sigma, f_0, f_2 \to \pi\pi, K\overline{K}, \eta\eta$

- ☐ **Briceño et al. 2017**

$\sigma \to \pi\pi$

- ☐ Prelovsek et al. 2010
- ☐ Fu 2013
- ☐ Wakayama 2015
- ☐ Howarth and Giedt 2017
- ☐ Briceño et al. 2017
- ☐ Guo et al. 2018

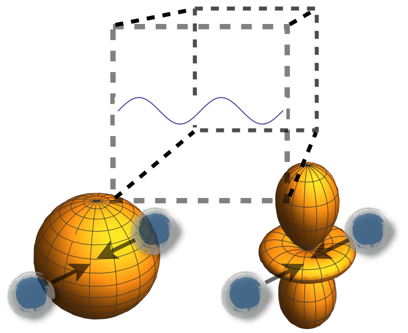


See the recent review by
Briceño, Dudek and Young

Coupled channels

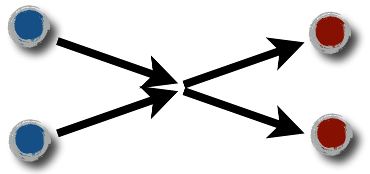
☐ The cubic volume mixes different partial waves…

$$\text{e.g.}\quad \begin{array}{c} K\pi \to K\pi \\ \vec{P} \neq 0 \end{array} \longrightarrow \det\left[\begin{pmatrix}\mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1}\end{pmatrix} + \begin{pmatrix}F_{ss} & F_{sp} \\ F_{ps} & F_{pp}\end{pmatrix}\right] = 0$$



…as well as different flavor channels…

$$\text{e.g.}\quad \begin{array}{c} a = \pi\pi \\ b = K\overline{K} \end{array} \longrightarrow \det\left[\begin{pmatrix}\mathcal{K}_{a\to a} & \mathcal{K}_{a\to b} \\ \mathcal{K}_{b\to a} & \mathcal{K}_{b\to b}\end{pmatrix}^{-1} + \begin{pmatrix}F_a & 0 \\ 0 & F_b\end{pmatrix}\right] = 0$$



☐ Workflow…

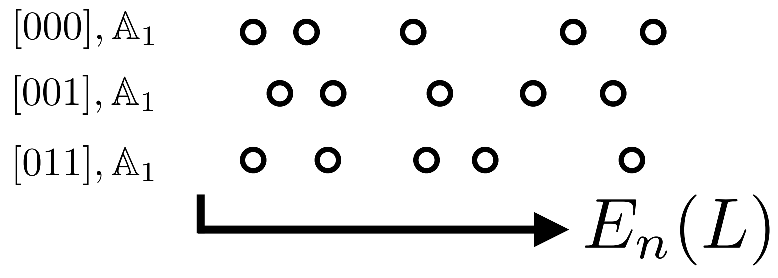
Correlators with a large operator basis

$$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$$

Reliably extract finite-volume energies

$$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$$

Vary L and P to recover a dense set of energies



hadspec

Identify a broad list of K-matrix parametrizations

polynomials and poles

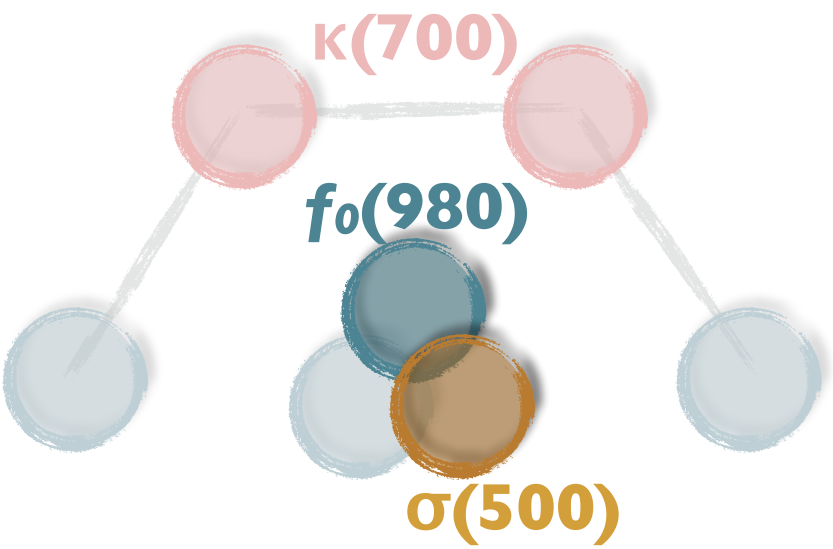
EFT based

dispersion theory based

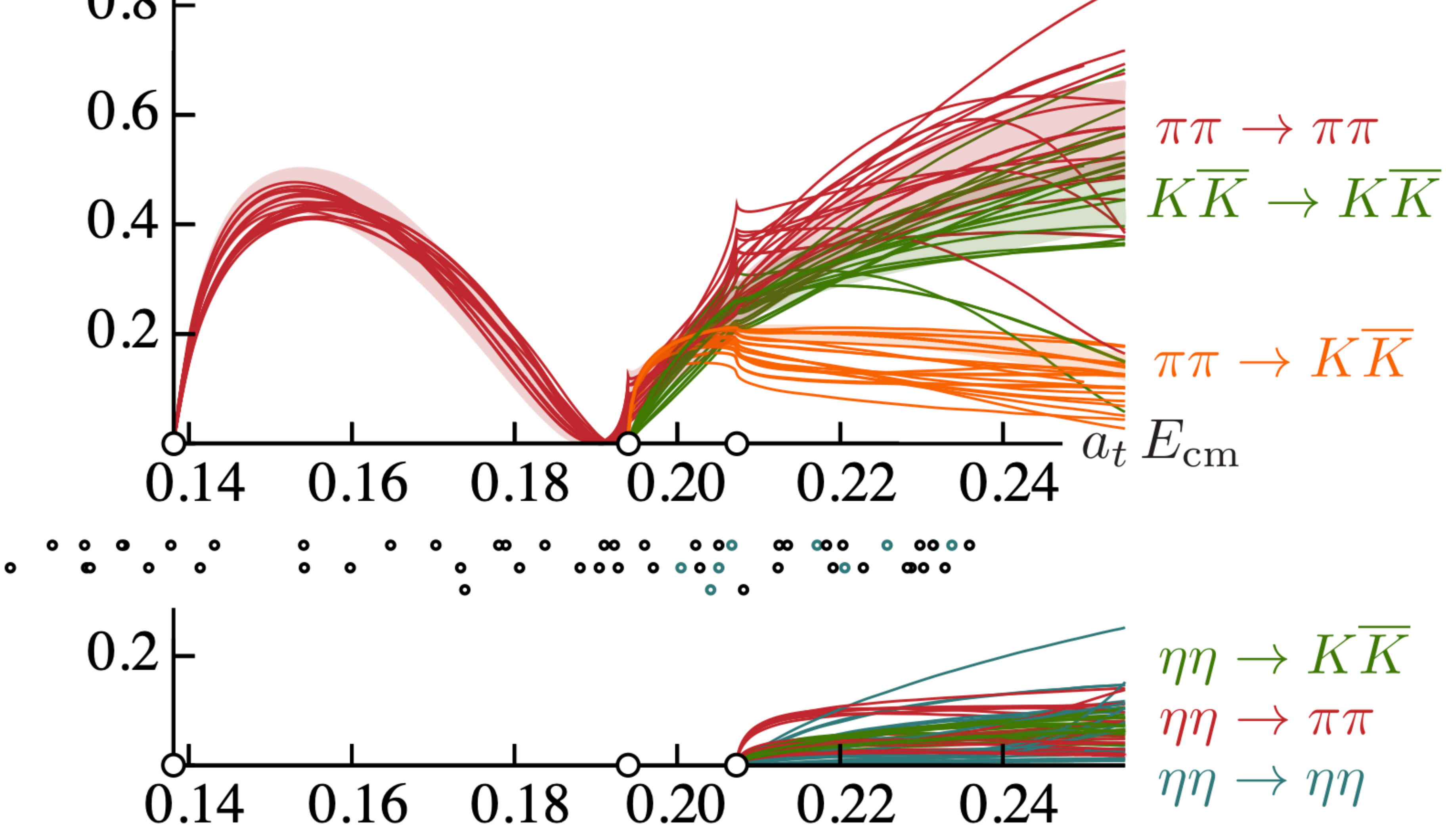
Perform global fits to the finite-volume spectrum

$$\sigma, f_0 \to \pi\pi, K\overline{K}, \eta\eta$$

$$I^G(J^{PC}) = 0^+(0^{++})$$



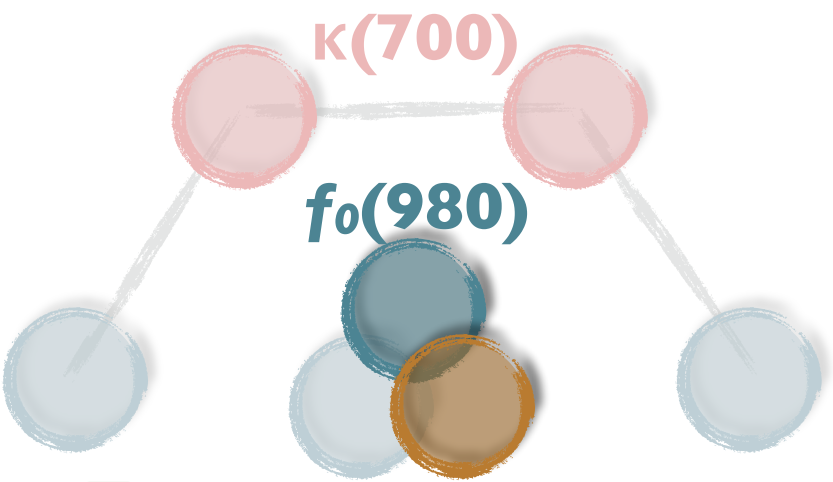
$\rho_i \rho_j |t_{ij}|^2$

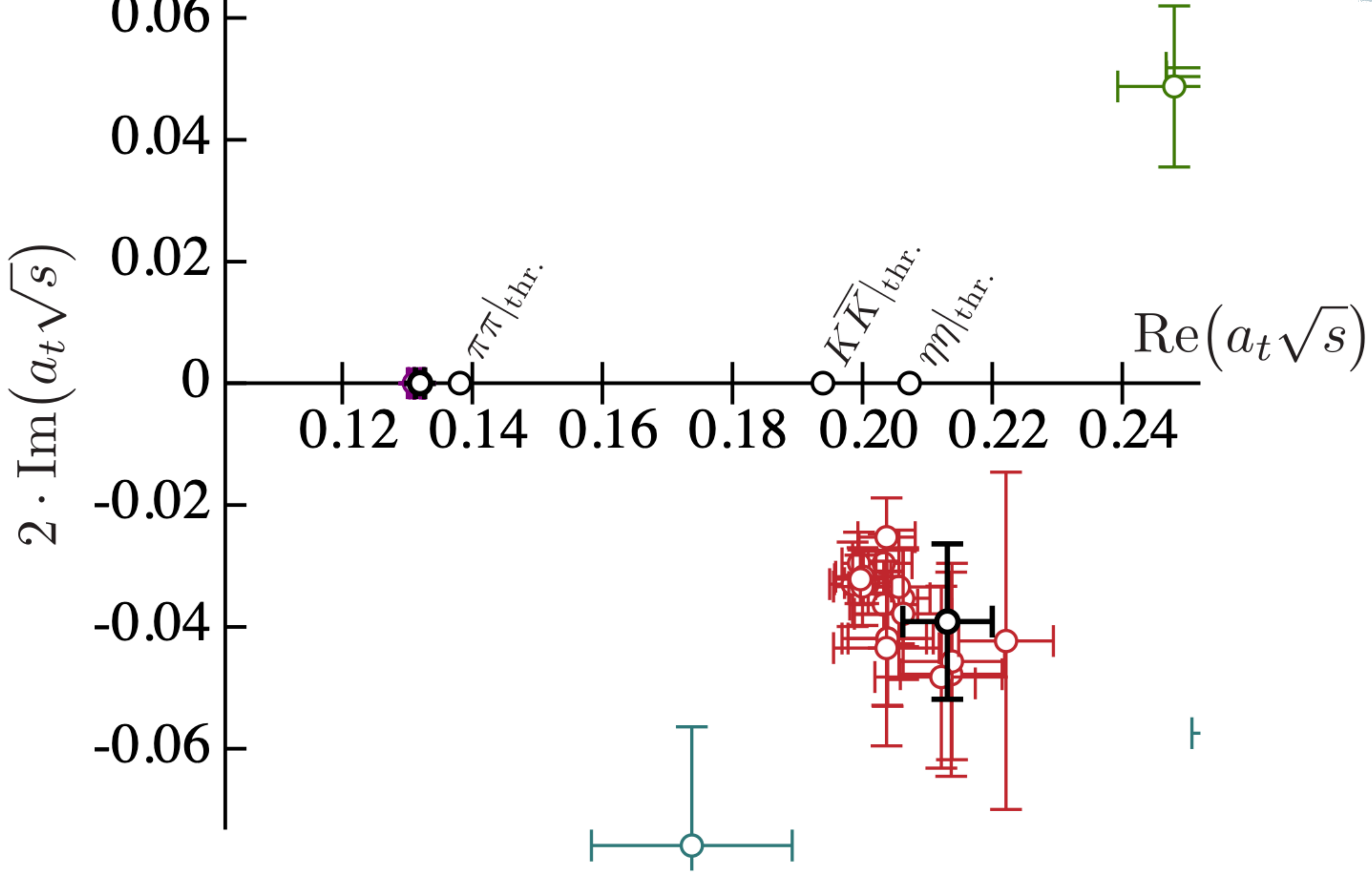


- Briceno et al., Phys.Rev. **D** 97 (2018) 5, 054513 •

$$\sigma, f_0 \to \pi\pi, K\overline{K}, \eta\eta$$

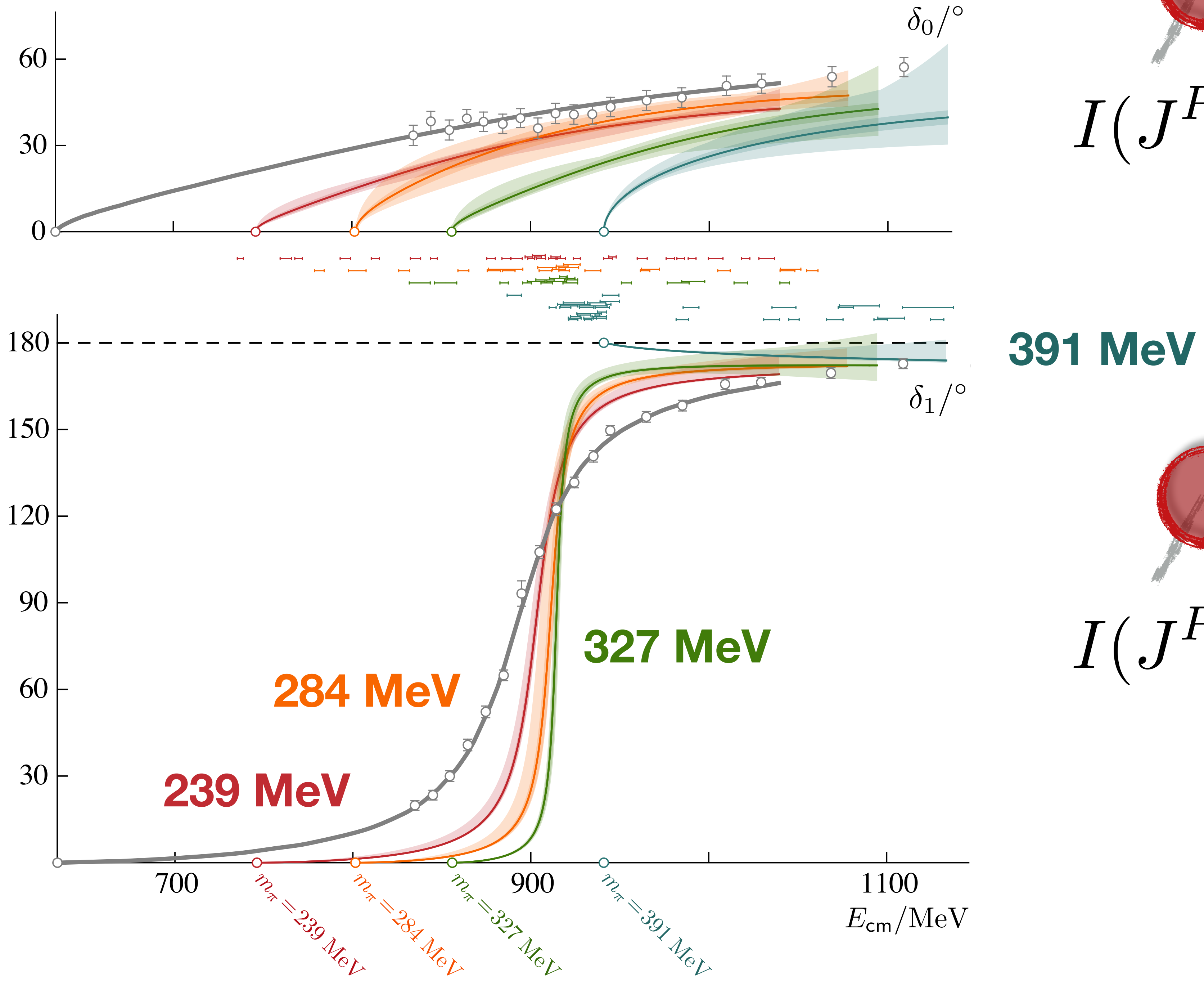
$$I^G(J^{PC}) = 0^+(0^{++})$$

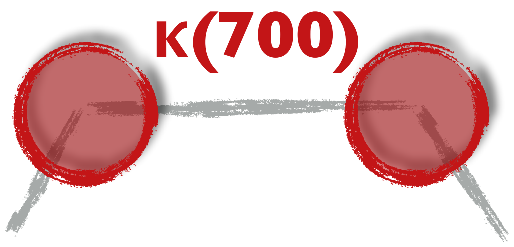




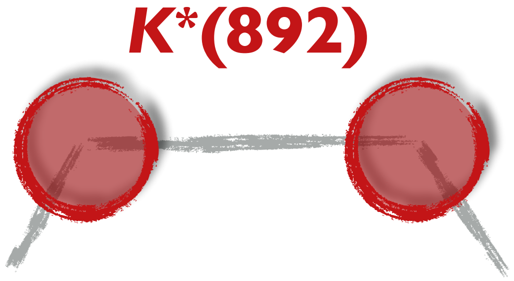
- Briceno et al., Phys.Rev. **D** 97 (2018) 5, 054513

$\kappa, K^* \to K\pi$





$$I(J^P) = 1/2(0^+)$$

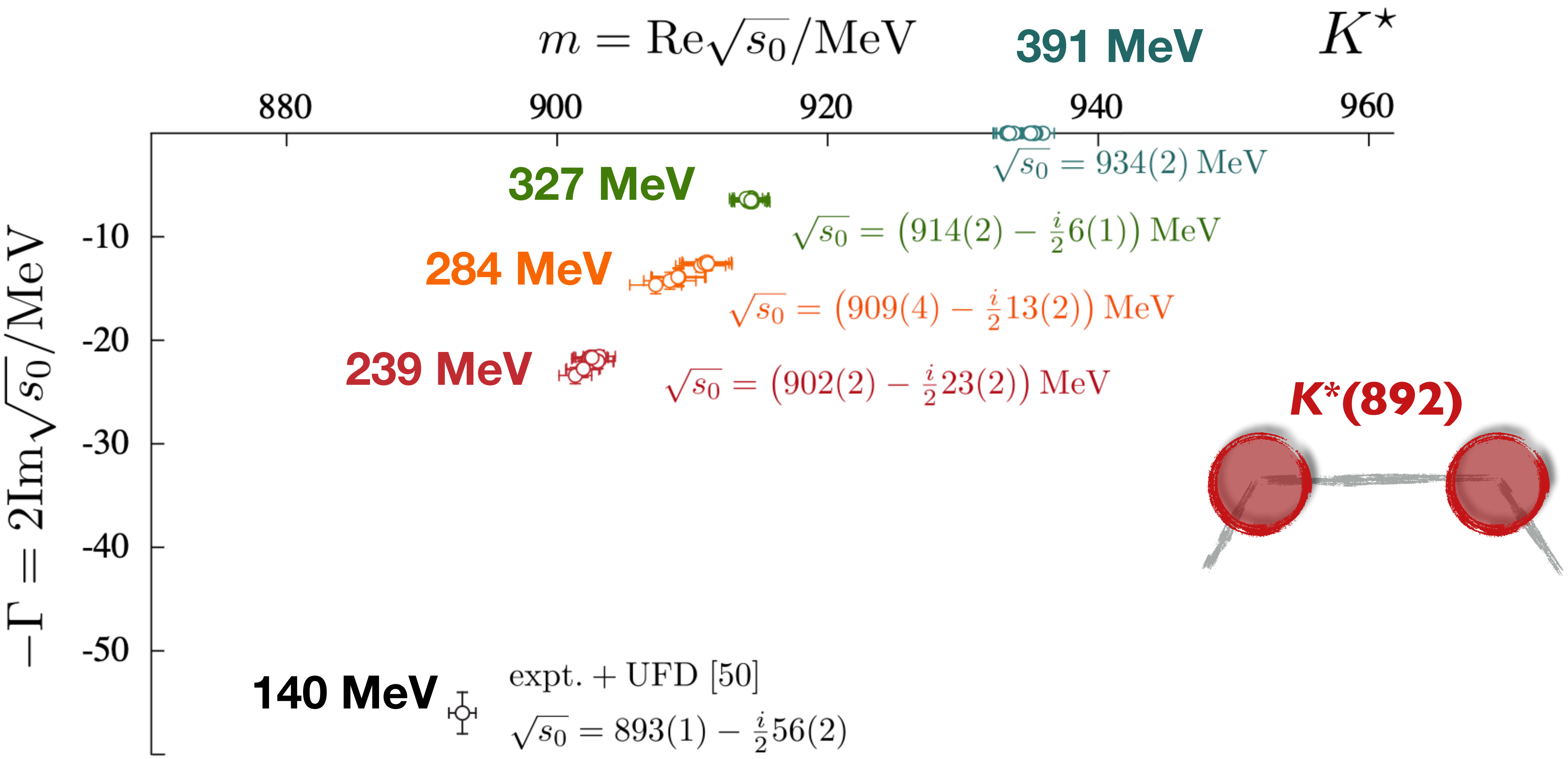


$$I(J^P) = 1/2(1^-)$$

- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002* •

$\kappa, K^* \to K\pi$

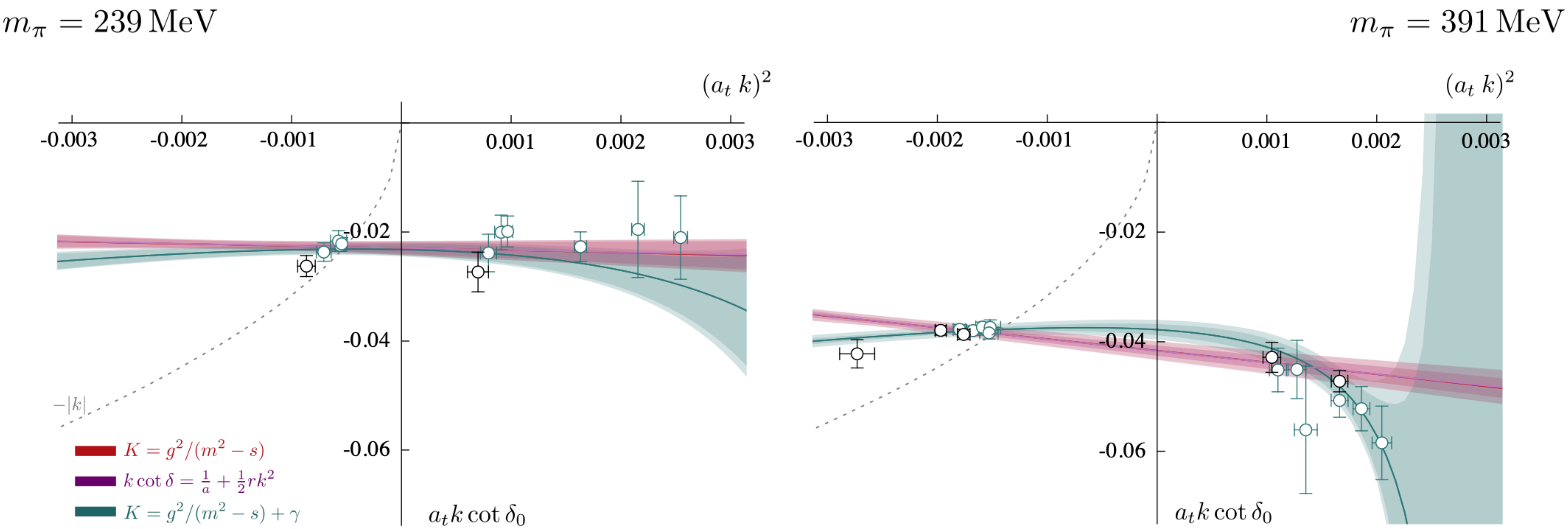
$$I(J^P) = 1/2(1^-)$$



- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002*

$D^*_{s0}(2317) \to DK$

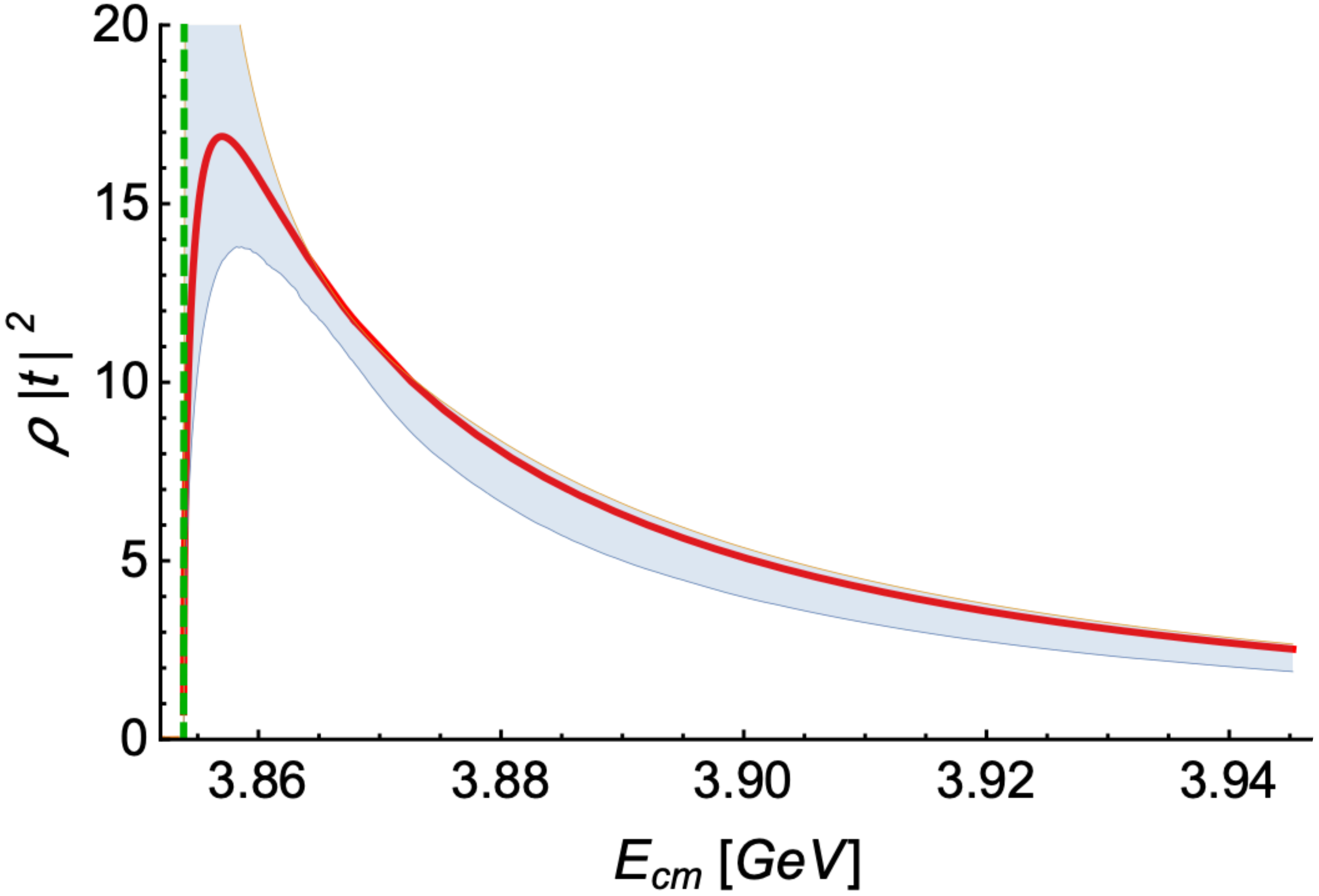
$I(J^P) = 0(0^+)$

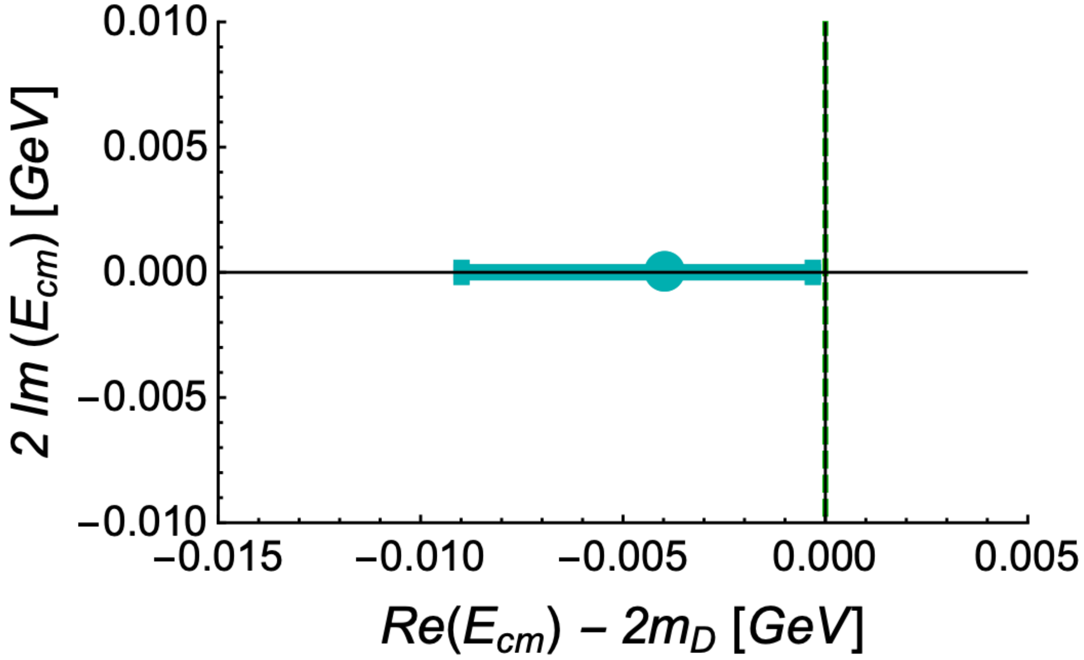


- Cheung et al. (Aug 14, 2020), *2008.06432 [hep-lat]*

$$D\bar{D}, D_s\bar{D}_s$$

$$J^{PC} = 0^{++}, 2^{++}$$

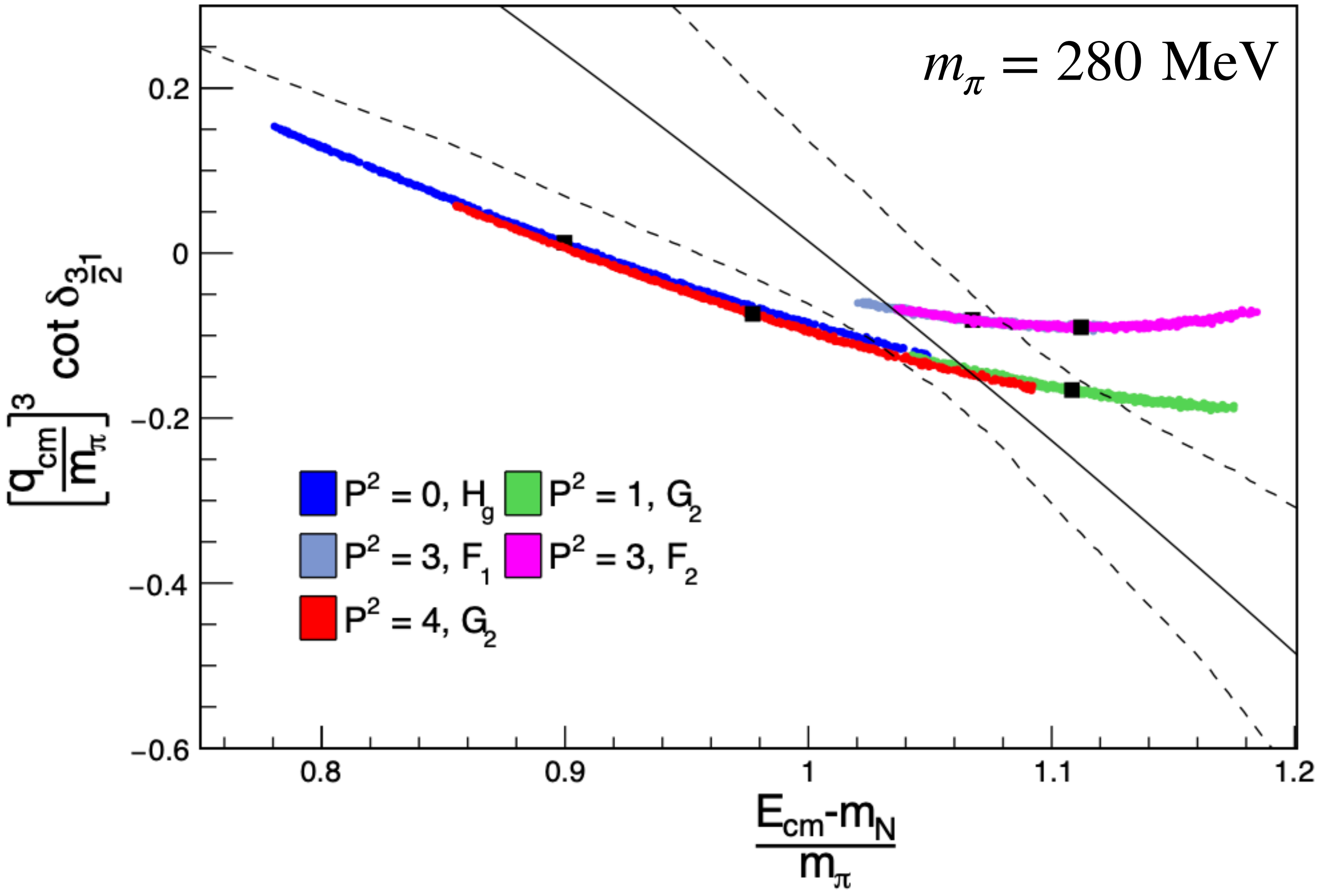




- Prelovsek et al. (November 4, 2020), *2011.02542 [hep-lat]* •

$\Delta(1232) \to N\pi$

$I(J^P) = 3/2(3/2^+)$



$$\frac{m_\Delta}{m_\pi} = 4.734(56), \qquad g^{\mathrm{BW}}_{\Delta N\pi} = 19.0(7.4),$$

$$(m_\pi a_{\frac{3}{2}2})^{-5} = 0.00(10), \quad (m_\pi a_{\frac{5}{2}2})^{-5} = 0.00(12),$$

$$\chi^2/\mathrm{d.o.f.} = 4.17.$$

• Andersen et al., Phys.Rev. **D** 97 (2018) 1, 014506 •

Five-day outline

- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} volume effects
  - ☑ Mass, matrix element in $\lambda\phi^4, g\phi^3$
  - ☑ Pion mass
  - ☑ LO-HVP for $(g-2)_\mu$
  - ☑ Bethe-Salpeter kernel
- ☑ $2 \to 2$ finite-volume (f.v.) formalism
  - ☑ Scattering basics
  - ☑ Derivation
  - ☑ Generalizations
  - ☑ Applications
- ☐ $(1+)\mathscr{J} \to 2$ f.v. formalism
  - ☐ Derivation
  - ☐ Expansions
  - ☐ Applications
- ☐ $2+\mathscr{J} \to 2$ f.v. formalism
  - ☐ Derivation
  - ☐ Testing the result
  - ☐ Numerical explorations
- ☐ Non-local matrix elements
  - ☐ Derivation
  - ☐ Applications
- ☐ $3 \to 3$ f.v. formalism
  - ☐ New complications
  - ☐ Derivation ($E_n(L)$ to $\mathscr{K}_{\text{df},3}$)
  - ☐ Integral equations ($\mathscr{K}_{\text{df},3}$ to $\mathscr{M}_3$)
  - ☐ Testing the result
  - ☐ Numerical explorations/calculations
- ☐ Finite-volume QED + QCD
  - ☐ Different formalisms
  - ☐ Basic examples