UK Research and Innovation

QCD (and QED) in a Euclidean finite volume

EuroPLEx Summer School 2021

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THE UNIVERSITY of EDINBURGH

Five-day outline

- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} volume effects
  - ☑ Mass, matrix element in $\lambda\phi^4, g\phi^3$
  - ☑ Pion mass
  - ☑ LO-HVP for $(g-2)_\mu$
  - ☑ Bethe-Salpeter kernel
- ☑ $2 \to 2$ finite-volume (f.v.) formalism
  - ☑ Scattering basics
  - ☑ Derivation
  - ☑ Generalizations
  - ☑ Applications
- ☐ $(1+)\mathscr{J} \to 2$ f.v. formalism
  - ☐ Derivation
  - ☐ Expansions
  - ☐ Applications
- ☐ $2 + \mathscr{J} \to 2$ f.v. formalism
  - ☐ Derivation
  - ☐ Testing the result
  - ☐ Numerical explorations
- ☐ Non-local matrix elements
  - ☐ Derivation
  - ☐ Applications
- ☐ $3 \to 3$ f.v. formalism
  - ☐ New complications
  - ☐ Derivation ($E_n(L)$ to $\mathscr{K}_{\text{df},3}$)
  - ☐ Integral equations ($\mathscr{K}_{\text{df},3}$ to $\mathscr{M}_3$)
  - ☐ Testing the result
  - ☐ Numerical explorations/calculations
- ☐ Finite-volume QED + QCD
  - ☐ Different formalisms
  - ☐ Basic examples

Five-day outline

- [x] Warm-up and definitions
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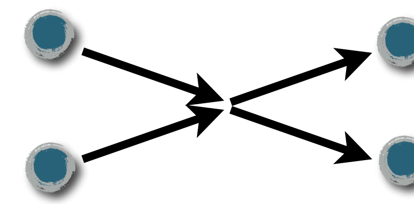
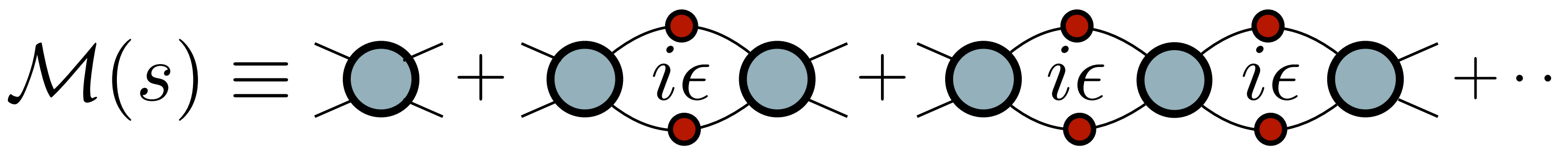
  - [ ] Basic examples

$2 \to 2$ analyticity

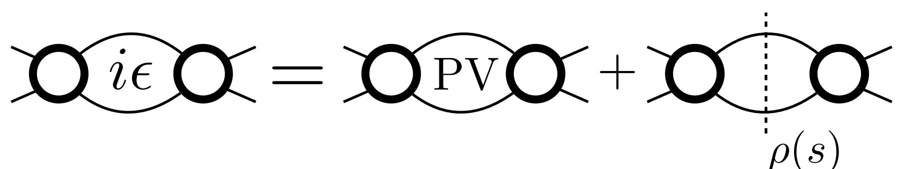
For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?





non-analytic:
on-shell particles = singularities

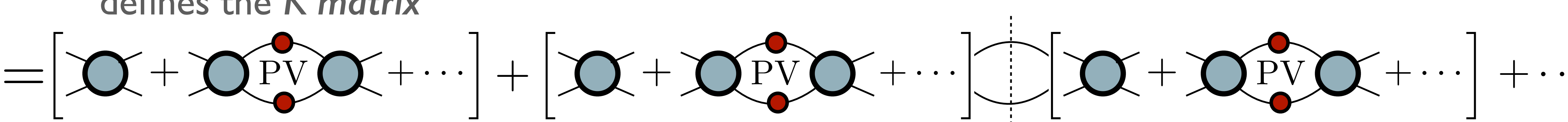
propagating pion



$\rho(s) \propto i\sqrt{s-(2m)^2}$

cutting rule

defines the *K matrix*

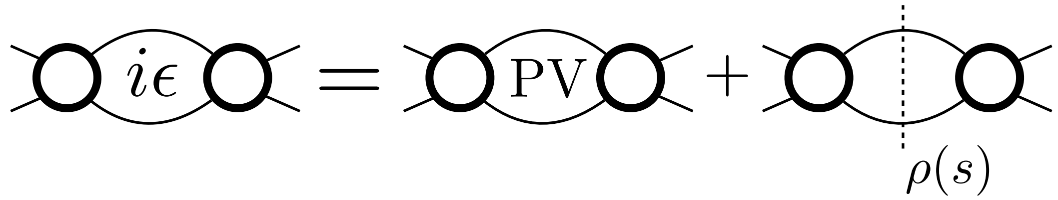


$\rho(s)$

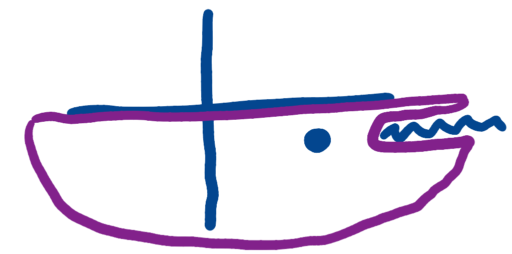
$$= \mathcal{K}(s) + \mathcal{K}(s)\rho(s)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - \rho(s)}$$

branch-cut singularity
$\sqrt{s-(2m)^2}$

Analyticity cutting rule



$$B \otimes \rho \otimes B = i\mathrm{Im}\Big[-i \int \frac{d^4k}{(2\pi)^4} B(s, \boldsymbol{p}, \boldsymbol{k}) D(k) D(P-k) B(s, \boldsymbol{k}, \boldsymbol{p}') \Big]$$

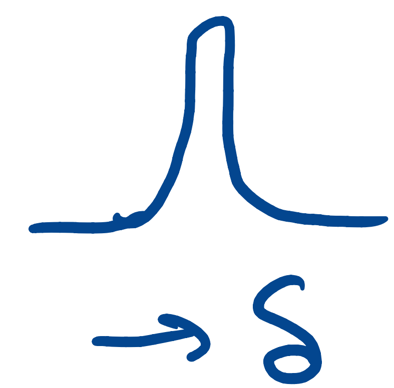


$$= i\mathrm{Im}\Big[\frac{1}{2} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{k}}} \frac{B(s, \boldsymbol{p}, \boldsymbol{k})\, B(s, \boldsymbol{k}, \boldsymbol{p}')}{(P-k)^2 + m^2 - \Sigma(P-k) - i\epsilon} \Big]$$

$$= -i\mathrm{Im}\Big[\frac{1}{2} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{(2\omega_{\boldsymbol{k}})^2} \frac{B(s, \boldsymbol{p}, \boldsymbol{k})\, B(s, \boldsymbol{k}, \boldsymbol{p}')}{E - 2\omega_{\boldsymbol{k}} + i\epsilon} \Big]$$

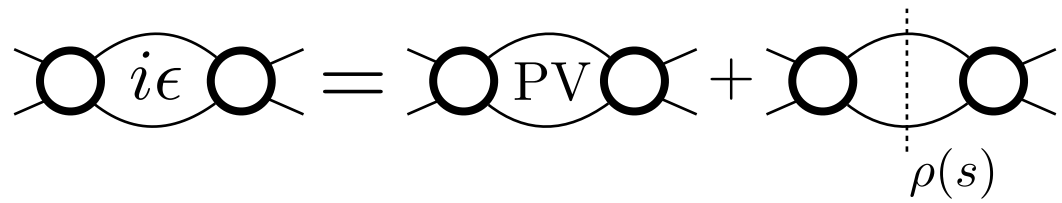
$$\mathrm{Im} \frac{1}{x + i\epsilon} = -\frac{\epsilon}{x^2 + \epsilon^2}$$

$$= i\pi \frac{1}{2} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{(2\omega_{\boldsymbol{k}})^2} B_{\mathsf{os}}(s, \hat{\boldsymbol{p}}, \hat{\boldsymbol{k}})\, B_{\mathsf{os}}(s, \hat{\boldsymbol{k}}, \hat{\boldsymbol{p}}') \delta(E - 2\omega_{\boldsymbol{k}})$$



$$= i\pi \mathcal{Y}^*(\hat{\boldsymbol{p}}) \cdot B_{\mathsf{os}}(s) \cdot \frac{1}{2} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{\mathcal{Y}(\hat{\boldsymbol{k}}) \mathcal{Y}^*(\hat{\boldsymbol{k}})}{(2\omega_{\boldsymbol{k}})^2} \delta(E - 2\omega_{\boldsymbol{k}}) \cdot B_{\mathsf{os}}(s) \cdot \mathcal{Y}(\hat{\boldsymbol{p}}')$$

Analyticity cutting rule



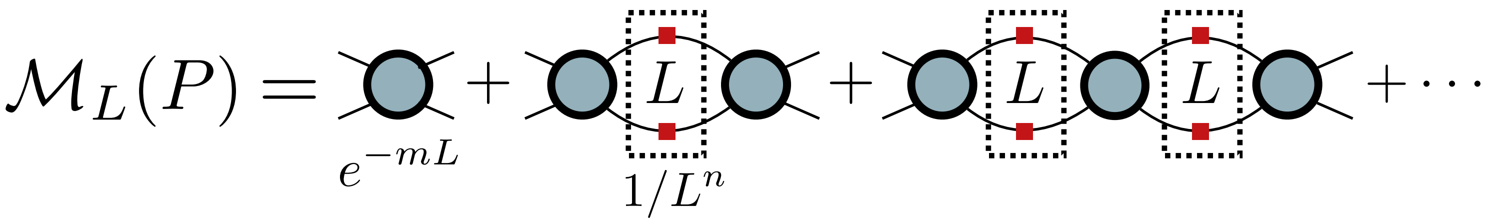
$$B \otimes \rho \otimes B = i\pi \mathcal{Y}^*(\hat{\boldsymbol{p}}) \cdot B_{\mathsf{os}}(s) \cdot \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{\mathcal{Y}(\hat{\boldsymbol{k}})\mathcal{Y}^*(\hat{\boldsymbol{k}})}{(2\omega_{\boldsymbol{k}})^2} \delta(E - 2\omega_{\boldsymbol{k}}) \cdot B_{\mathsf{os}}(s) \cdot \mathcal{Y}(\hat{\boldsymbol{p}}')$$

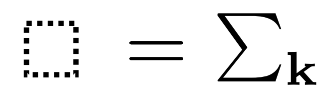
$$\frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{\mathcal{Y}(\hat{\boldsymbol{k}})\mathcal{Y}^*(\hat{\boldsymbol{k}})}{(2\omega_{\boldsymbol{k}})^2} \delta(E - 2\omega_{\boldsymbol{k}}) = \mathbb{I} \frac{4\pi}{2(2\pi)^3} \int_0^\infty dk k^2 \frac{\delta(E - 2\omega_k)}{4\omega_k^2}$$

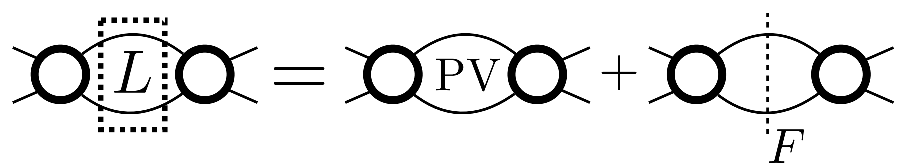
$$= \mathbb{I} \frac{\sqrt{1 - 4m^2/s}}{32\pi}$$

$2 \to 2$ in a finite volume

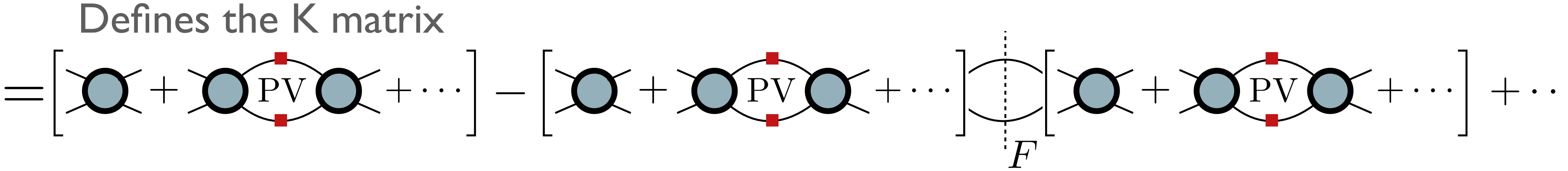
For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?



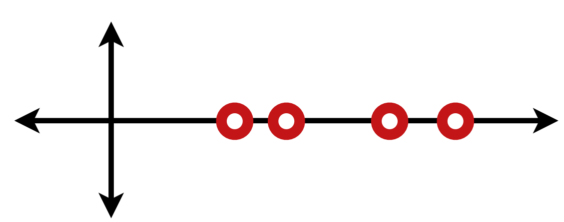




Defines the K matrix

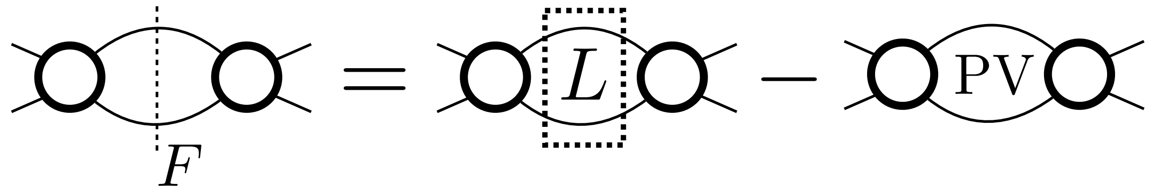


$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P, L)}$$



- Lüscher (1986)
- Kim, Sachrajda, Sharpe (2005)
- MTH, Sharpe (*coupled channels*, 2012)

Finite-volume cutting rule



$$B \otimes F \otimes B = -i\frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\right]\int\frac{dk^0}{2\pi}B(s,\boldsymbol{p},\boldsymbol{k})D(k)D(P-k)B(s,\boldsymbol{k},\boldsymbol{p}')$$

$$= -\frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\right]\frac{1}{2\omega_{\boldsymbol{k}}}\frac{B(s,\boldsymbol{p},\boldsymbol{k})\,B(s,\boldsymbol{k},\boldsymbol{p}')}{(P-k)^2+m^2-\Sigma(P-k)}\bigg|_{k^0=\omega_{\boldsymbol{k}}} + \mathcal{O}(e^{-mL})$$

$$= \frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\right]\frac{1}{(2\omega_{\boldsymbol{k}})^2}\frac{B_{\text{os}}(s,\hat{\boldsymbol{p}},\hat{\boldsymbol{k}})\,B_{\text{os}}(s,\hat{\boldsymbol{k}},\hat{\boldsymbol{p}}')}{E-2\omega_{\boldsymbol{k}}} + \mathcal{O}(e^{-mL})$$

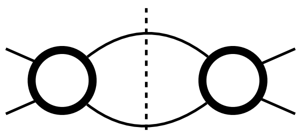
os = on shell

$$B_{\text{os}}(s,\hat{\boldsymbol{k}},\hat{\boldsymbol{p}}) = B(s,\sqrt{s/4-m^2}\,\hat{\boldsymbol{k}},\sqrt{s/4-m^2}\,\hat{\boldsymbol{p}})$$

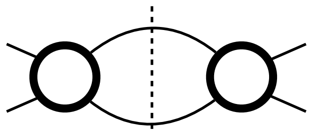
$$\frac{BB-(BB)_{os}}{E-2\omega_k} = \text{smooth}$$

$$= \mathcal{Y}^*(\hat{\boldsymbol{p}})\cdot B_{\text{os}}(s)\cdot\left[\frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\right]\frac{1}{(2\omega_{\boldsymbol{k}})^2}\frac{\mathcal{Y}(\hat{\boldsymbol{k}})\mathcal{Y}^*(\hat{\boldsymbol{k}})}{E-2\omega_{\boldsymbol{k}}}\right]\cdot B_{\text{os}}(s)\cdot\mathcal{Y}^*(\hat{\boldsymbol{p}}') + \mathcal{O}(e^{-mL})$$

Comparison of cuts



$\rho(s)$



F

$$\rho(s) = \mathbb{I}\frac{\sqrt{1-4m^2/s}}{32\pi}$$

$$F(E,L) = \frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right]\frac{1}{(2\omega_{\boldsymbol{k}})^2}\frac{\mathcal{Y}(\hat{\boldsymbol{k}})\mathcal{Y}^*(\hat{\boldsymbol{k}})}{E-2\omega_{\boldsymbol{k}}}$$

No volume dependence

Volume dependent

Both can be understood as matrices in $\ell m, \ell' m'$

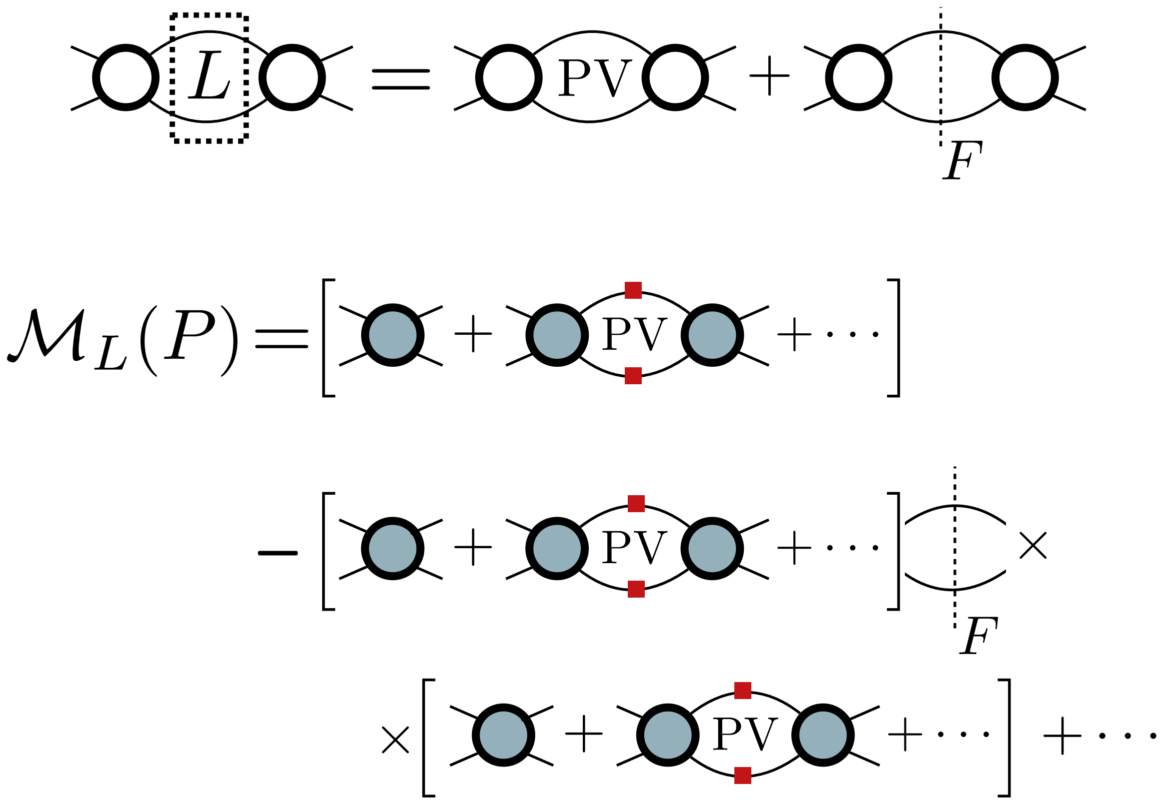
proportional to the identity

on- and off-diagonal elements

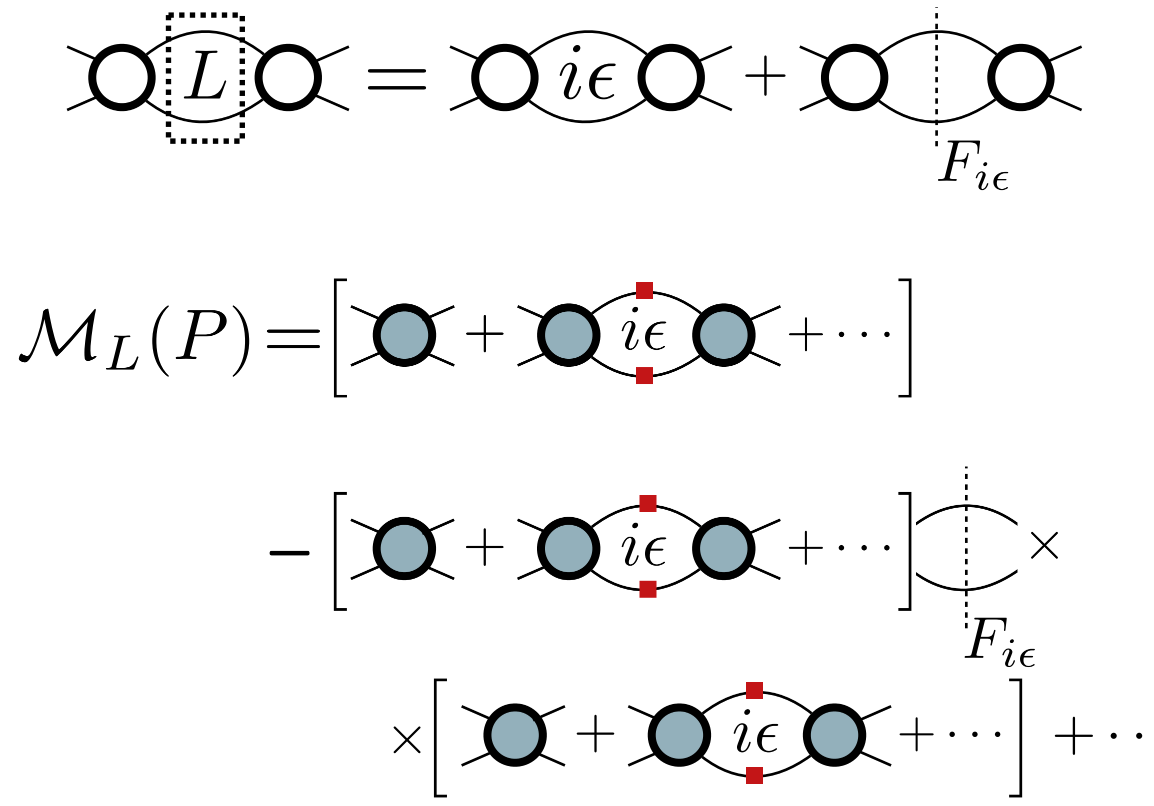
Exact difference between $i\epsilon$ and PV

Approximate difference between sum and PV

Pole prescription freedom



$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$



$$= \frac{1}{\mathcal{M}(s)^{-1} + F_{i\epsilon}(P,L)}$$

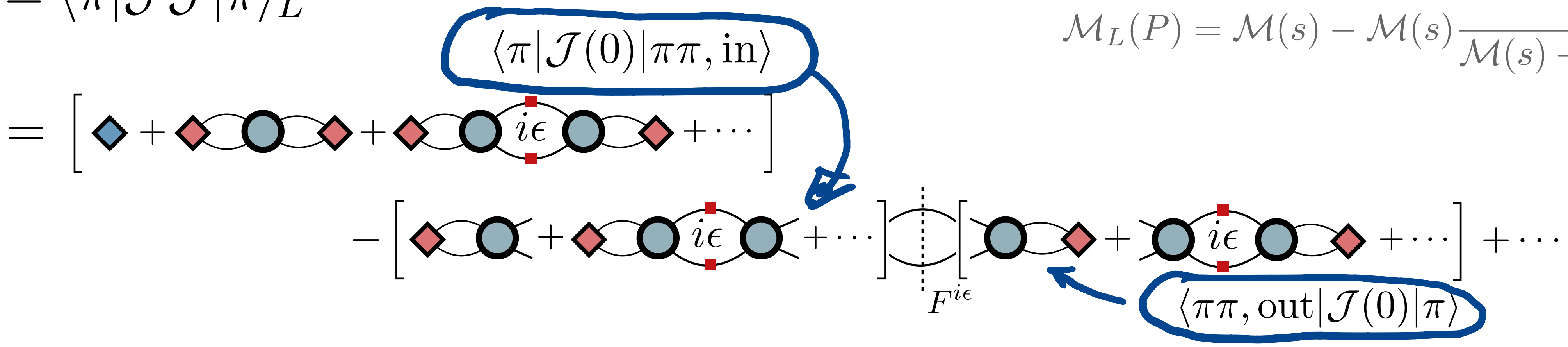
$$\mathcal{K}(s)^{-1} + F(P,L) = [\mathcal{K}(s)^{-1} - i\rho(s)] + [F(P,L) + i\rho(s)] = \mathcal{M}(s)^{-1} + F_{i\epsilon}(P,L)$$

$1 + \mathcal{J} \to 2$

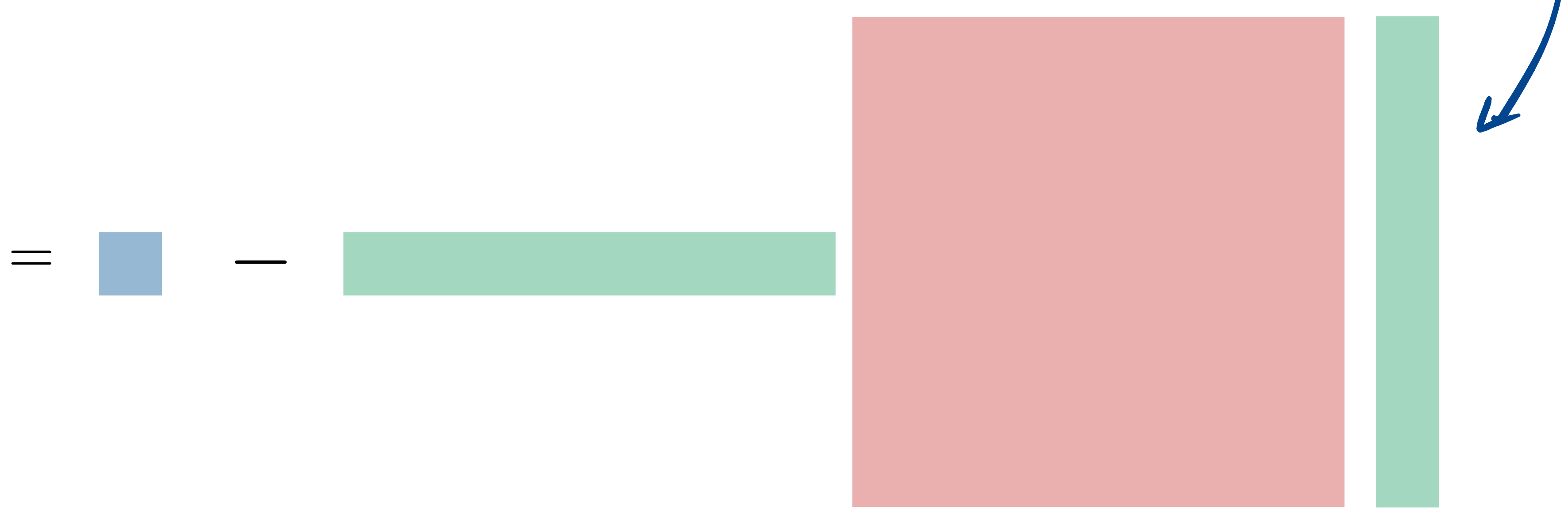
Instead of this…

$$\mathcal{M}_L(P) = \mathcal{M}(s) - \mathcal{M}(s)\frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}}\mathcal{M}(s)$$

$$C_L(P) = \langle \pi | \mathcal{J}\mathcal{J} | \pi \rangle_L$$



$$= C_\infty^{i\epsilon}(s) - A_{\text{in}}^{i\epsilon}(s)\frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}}A_{\text{out}}^{i\epsilon}(s)$$



Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016)

$1 + \mathcal{J} \to 2$

$$C_L(P) = C_\infty^{i\epsilon}(s) - A_{\text{in}}^{i\epsilon}(s) \frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} A_{\text{out}}^{i\epsilon}(s)$$

Useful, since… $\lim_{E \to E_n(L)} [E - E_n(L)] C_L(P) \;\propto\; |\langle n, L | \mathcal{J} | \pi, L \rangle|^2$

$$C_L(P) = \int_L d^4x \, e^{-iPx} \langle \pi | \mathcal{J}(x) \mathcal{J}(0) | \pi \rangle$$

$$\lim_{E \to E_n(L)} (E - E_n(L)) \int_0^\infty dt \, \langle \pi | \tilde{\mathcal{J}}(0,\vec{p}) \, e^{-i\hat{H}t - \varepsilon t} \sum_n | n, \vec{p}, L \rangle \langle n, \vec{p}, L | \mathcal{J}(0) | \pi \rangle e^{iEt}$$

$$= -L^3 \, |\langle n, \vec{p}, L | \mathcal{J}(0) | \pi \rangle|^2$$

Crucial information = residue at the pole

$$\mathcal{R}(P,L) = \lim_{E \to E_n(L)} \frac{E - E_n(L)}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} = \frac{1}{\mu'(E)} \mathbf{v}^T \mathbf{v}$$

Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016)

$\mathscr{R}$ is rank one

$$C_L(P) = C_\infty^{i\epsilon}(s) - A_{\text{in}}^{i\epsilon}(s) \frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} A_{\text{out}}^{i\epsilon}(s)$$

Crucial information = residue at the pole

$$\mathcal{R}(P,L) = \lim_{E \to E_n(L)} \frac{E - E_n(L)}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} = \frac{1}{\mu'(E)} \mathbf{v}^T \mathbf{v}$$

$$= \lim_{E \to E_n(L)} (E - E_n(L)) \left[\frac{1}{\mu(E)} v v^T \right]$$

$$= \frac{1}{\mu'(E)} v v^T$$

Putting it all together…

Finite-volume matrix element

$$C_L(P) = \int_L d^4x\, e^{-iPx}\, \langle \pi | \mathrm{T}\{\mathcal{J}(x)\mathcal{J}(0)\} | \pi \rangle_L$$

Insert a complete set finite-volume of states

$$\xrightarrow[E \to E_n]{} -\frac{L^3 \langle \pi | \mathcal{J}(0) | n, L \rangle \langle n, L | \mathcal{J}(0) | \pi \rangle}{E - E_n(L)}$$

Infinite-volume matrix element

$$C_L(P) = C_\infty^{i\epsilon}(s) - A_{\text{in}}^{i\epsilon}(s) \frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} A_{\text{out}}^{i\epsilon}(s)$$

$$\xrightarrow[E \to E_n]{} -\frac{\langle \pi | \mathcal{J}(0) | \pi\pi, \mathrm{in} \rangle\, \mathcal{R}(E, \boldsymbol{P}, L)\, \langle \pi\pi, \mathrm{out} | \mathcal{J}(0) | \pi \rangle}{E - E_n(L)}$$

Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016)

Transition amplitudes

$$\sqrt{2M_\pi}L^3|\langle n, L|\mathcal{J}(0)|\pi\rangle| = \frac{1}{\sqrt{\mu'(E)}}\left|\mathbf{v}_\alpha\langle\alpha,\mathrm{out}|\mathcal{J}(0)|\pi\rangle\right|$$

$$\mathcal{R}(P,L) = \lim_{E\to E_n(L)} \frac{E-E_n(L)}{\mathcal{M}(s)+F^{i\epsilon}(P,L)^{-1}} = \frac{1}{\mu'(E)}\mathbf{v}^T\mathbf{v}$$

α can run over angular momentum and flavor ($\pi\pi, K\overline{K}$)

result really holds for any single incoming hadron, any local current

generalization of Lellouch-Lüscher formalism for $K\to\pi\pi$

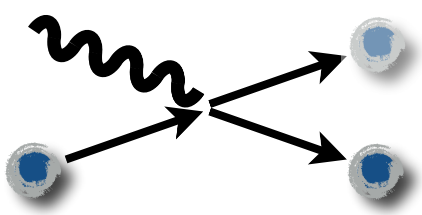
application requires truncation in angular momentum space (as did LL formalism)

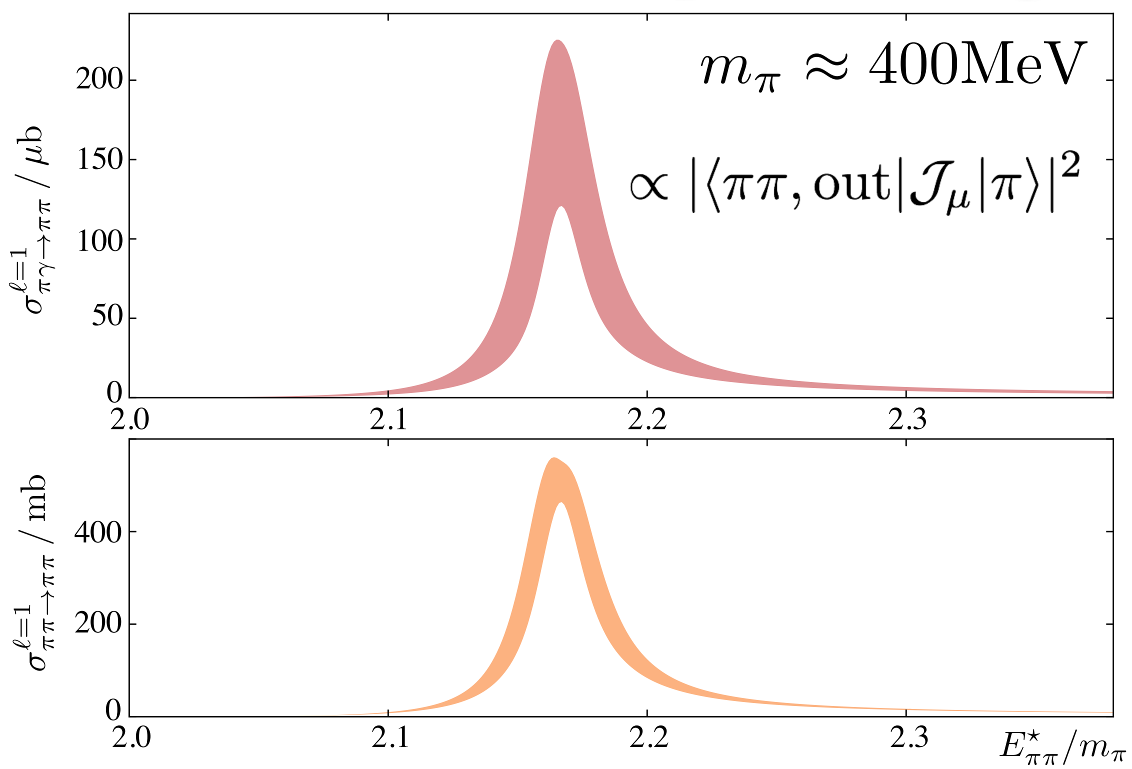
approx. $\mathcal{M}_\ell$, K_ℓ, δ_ℓ $= 0$ for some $\ell > \ell_{max}$

Transition amplitudes

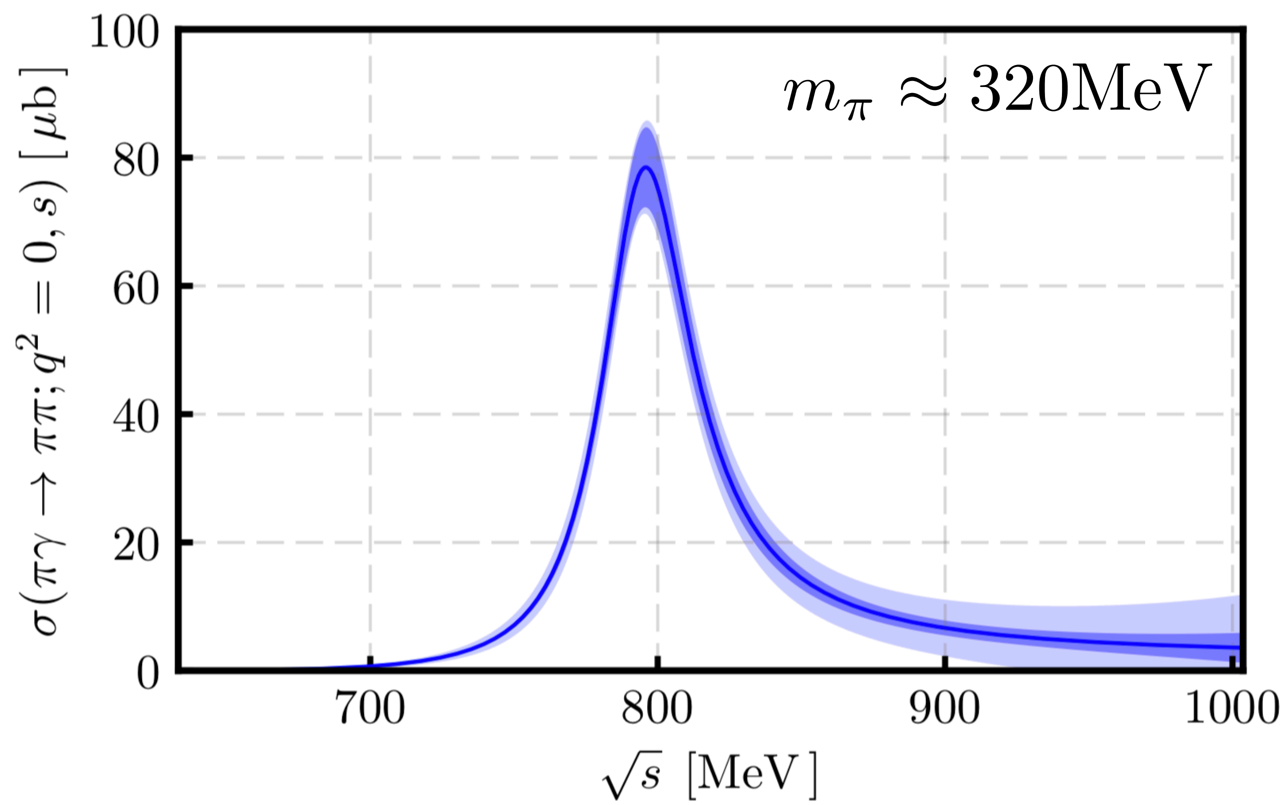
$$\sqrt{2M_\pi}L^3|\langle n, L|\mathcal{J}(0)|\pi\rangle| = \frac{1}{\sqrt{\mu'(E)}}\left|\mathbf{v}_\alpha\langle\alpha, \mathrm{out}|\mathcal{J}(0)|\pi\rangle\right|$$

$$\mathcal{R}(P,L) = \lim_{E\to E_n(L)} \frac{E - E_n(L)}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} = \frac{1}{\mu'(E)}\mathbf{v}^T\mathbf{v}$$

$\langle\pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 



Briceño et. al., Phys. Rev. D93, 114508 (2016)

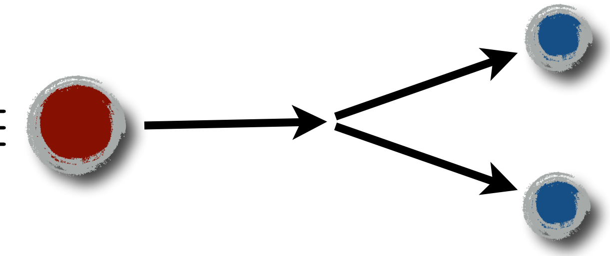


Alexandrou et. al., Phys. Rev. D98, 074502 (2018)

Same idea, many contexts

Kaon decay

$$\langle \pi\pi, \mathrm{out}|\mathcal{H}|K\rangle \equiv$$

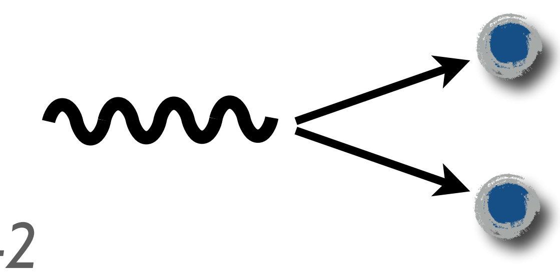


Implementation by RBC/UKQCD collaboration

Lellouch, Lüscher (2001) • Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005)

Time-like form factors

$$\langle \pi\pi,\ \mathrm{out}|\mathcal{J}_\mu|0\rangle \equiv$$

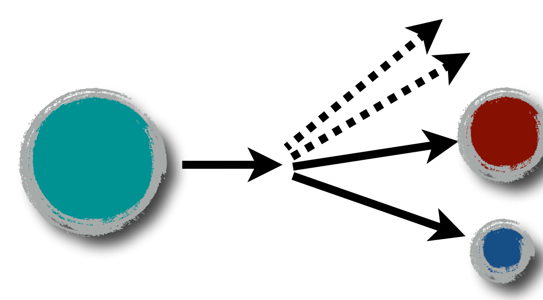


Relevant for muon HVP contribution to muon g-2

Meyer (2011)

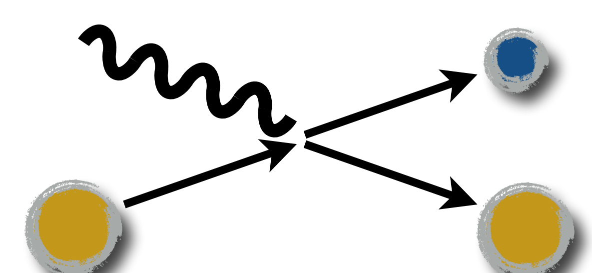
Resonance transition amplitudes

$$\langle K\pi, \mathrm{out}|\mathcal{J}_{\alpha\beta}|B\rangle \equiv$$



Particles with spin

$$\langle N\pi, \mathrm{out}|\mathcal{J}_\mu|N\rangle \equiv$$

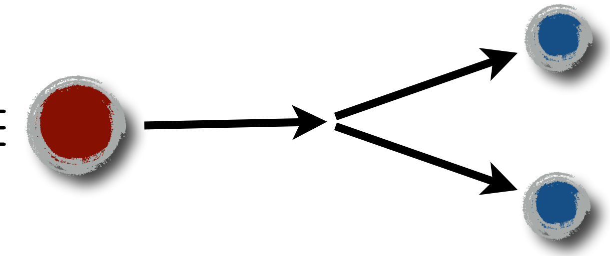


Agadjanov *et al*. (2014) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016)

Really the same?

Kaon decay

$$\langle \pi\pi, \mathrm{out}|\mathcal{H}|K\rangle \equiv$$



Implementation by RBC/UKQCD collaboration

Lellouch, Lüscher (2001) • Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005)

$$|A|^2 = 8\pi \left\{ q\frac{\partial\phi}{\partial q} + k\frac{\partial\delta_0}{\partial k} \right\}_{k=k_\pi} \left(\frac{m_K}{k_\pi}\right)^3 |M|^2$$

$$T(K \to \pi\pi) = A\mathrm{e}^{i\delta_0} \tag{2.4}$$

with A real and δ_0 the S-wave scattering phase shift of the outgoing pion state. The decay rate is then given by the usual expression

$$\Gamma = \frac{k_\pi}{16\pi m_K^2}|A|^2, \qquad k_\pi \equiv \tfrac{1}{2}\sqrt{m_K^2 - 4m_\pi^2}, \tag{2.5}$$

proportional to the pion momentum k_π in the centre-of-mass frame.

Relation to Lellouch-Lüscher

$$\mathcal{R}(E_n(L), L)^{-1} = \partial_E \left[F_{i\epsilon}(E,L)^{-1} + \mathcal{M}(E)\right]\Big|_{E=E_n(L)}$$

This multiplies the F.V. matrix element

$$= -\mathcal{M}(E)^2\, \partial_E \left[F_{i\epsilon}(E,L) + \mathcal{M}(E)^{-1}\right]\Big|_{E=E_n(L)}$$

$$= -\left(\frac{16\pi E}{p}\right)^2 e^{2i\delta(E)} \sin^2\delta(E)\, \partial_E \left[F(E,L) + \mathcal{K}(E)^{-1}\right]\Big|_{E=E_n(L)}$$

$$\frac{1}{p\cot\delta - ip} = \frac{1}{p}\frac{\sin\delta}{\cos\delta - i\sin\delta} = \frac{\sin\delta}{p} e^{i\delta}$$

$$= -\left(\frac{16\pi E}{p}\right) e^{2i\delta(E)} \sin^2\delta(E)\, \partial_E \left[\cot\phi(E,L) + \cot\delta(E)\right]\Big|_{E=E_n(L)}$$

$$= \frac{16\pi E}{p} e^{2i\delta(E)}\, \partial_E \left[\phi(E,L) + \delta(E)\right]\Big|_{E=E_n(L)}$$

$$q = \frac{\sqrt{E^2/4 - m^2}\, L}{2\pi}$$

$$= \frac{4\pi E^2}{p^3} e^{2i\delta(p)} \left[q\frac{\partial\phi(q)}{\partial q} + p\frac{\partial\delta(p)}{\partial p}\right]_{E=E_n(L)}$$

Relation to Lellouch-Lüscher

$$\mathcal{R}(E_n(L),L)^{-1} = \frac{4\pi E^2}{p^3} e^{2i\delta(p)} \left[q\frac{\partial\phi(q)}{\partial q} + p\frac{\partial\delta(p)}{\partial p} \right]_{E=E_n(L)}$$

$$\langle K|\mathcal{H}_W(0)|\pi\pi,\mathrm{in}\rangle\langle\pi\pi,\mathrm{out}|\mathcal{H}_W(0)|K\rangle = 2M_K\mathcal{R}^{-1}|\langle n,L|H_W(0)|K\rangle|^2$$

$$= \frac{2M_K\, 4\pi(M_K)^2}{p^3}\left[\dots \right]$$

$$|A|^2 = 8\pi \left\{ q\frac{\partial\phi}{\partial q} + k\frac{\partial\delta_0}{\partial k} \right\}_{k=k_\pi} \left(\frac{m_K}{k_\pi}\right)^3 |M|^2$$

Lellouch, Lüscher (2001) • Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005)

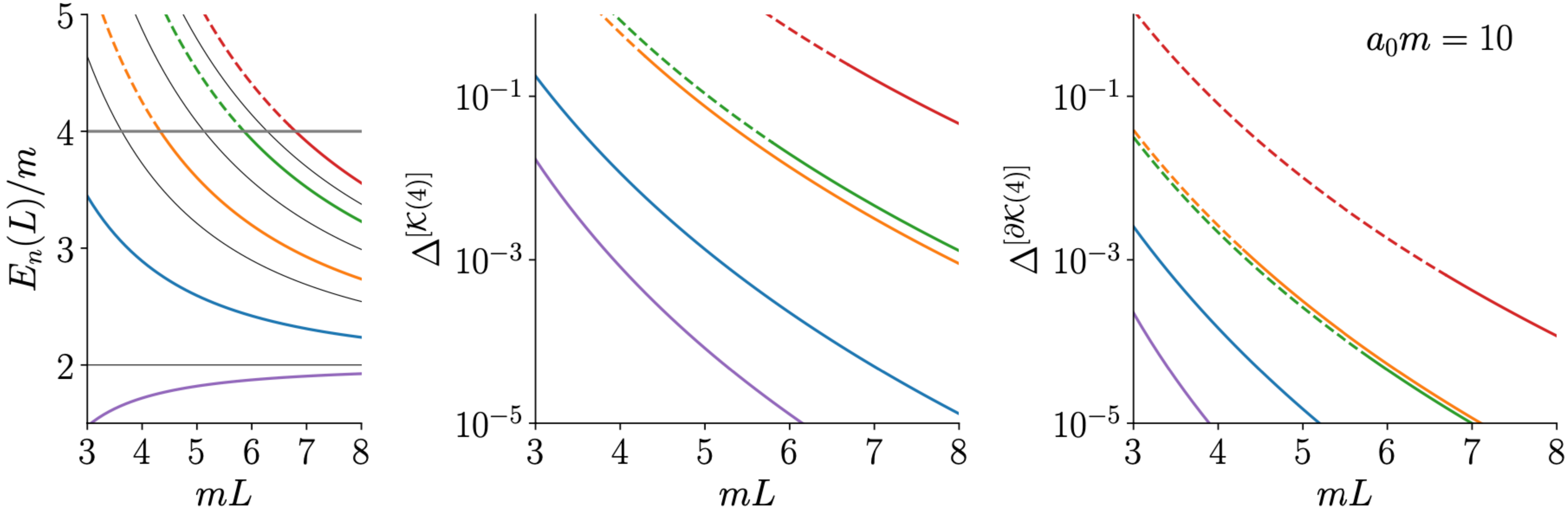
Angular momentum contamination

Even $K \to \pi\pi$ gets corrections from higher angular momenta

$$\Delta(E_n(L), L) = \frac{\mathcal{R}(E_n(L), L)^{-1}_{\ell_{\max}=4} - \mathcal{R}(E_n(L), L)^{-1}_{\ell_{\max}=0}}{\mathcal{R}(E_n(L), L)^{-1}_{\ell_{\max}=4}}$$

Present this as

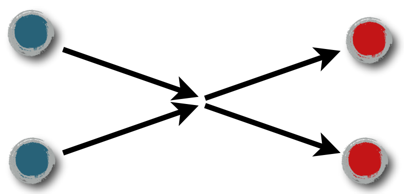
$$\Delta = \Delta^{[K(4)]}(2m)^8 \frac{\mathcal{K}^{(4)}}{p^8} + \Delta^{[\partial K(4)]}(2m)^9 \frac{\partial\mathcal{K}^{(4)}}{p^8}$$



MTH, T. Peterken *to appear*

D decays as a frontier

infinite-volume
scattering amplitudes

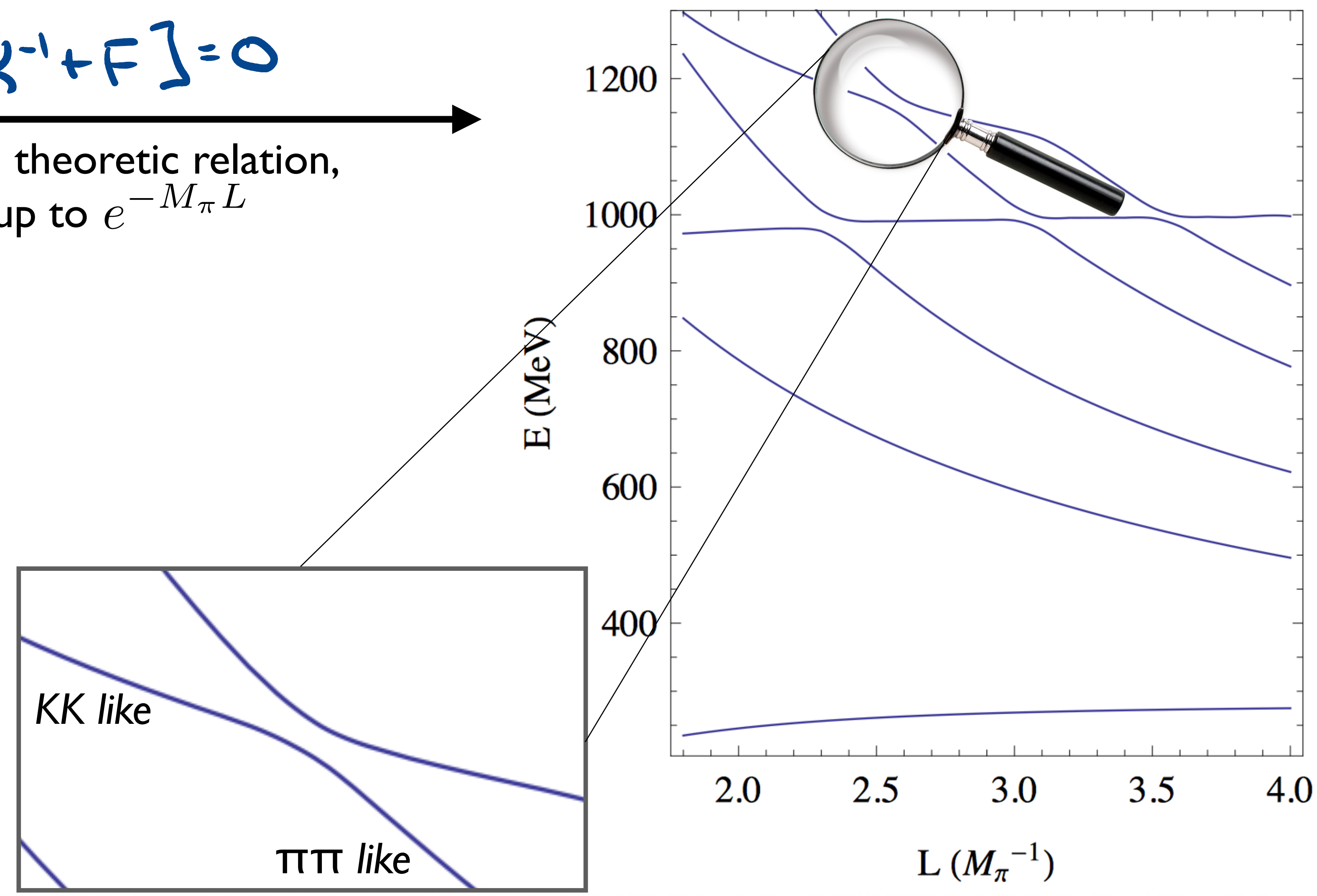


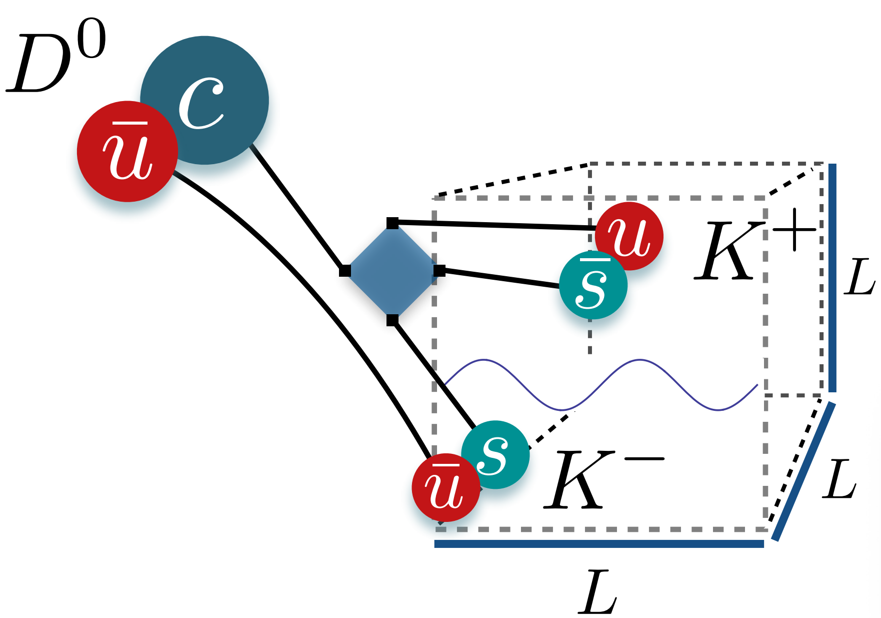
$\pi\pi \to K\overline{K}$

$\det[K^{-1}+F]=0$

know field theoretic relation,
exact up to $e^{-M_\pi L}$

finite-volume energies





$$|n, L\rangle = c_\pi^{(n)}(L)\,|\pi\pi, \text{out}\rangle + c_K^{(n)}(L)\,|K\overline{K}, \text{out}\rangle$$

path to physical amplitude

• MTH, Sharpe, *Phys.Rev.* **D**86 (2012) 016007 •

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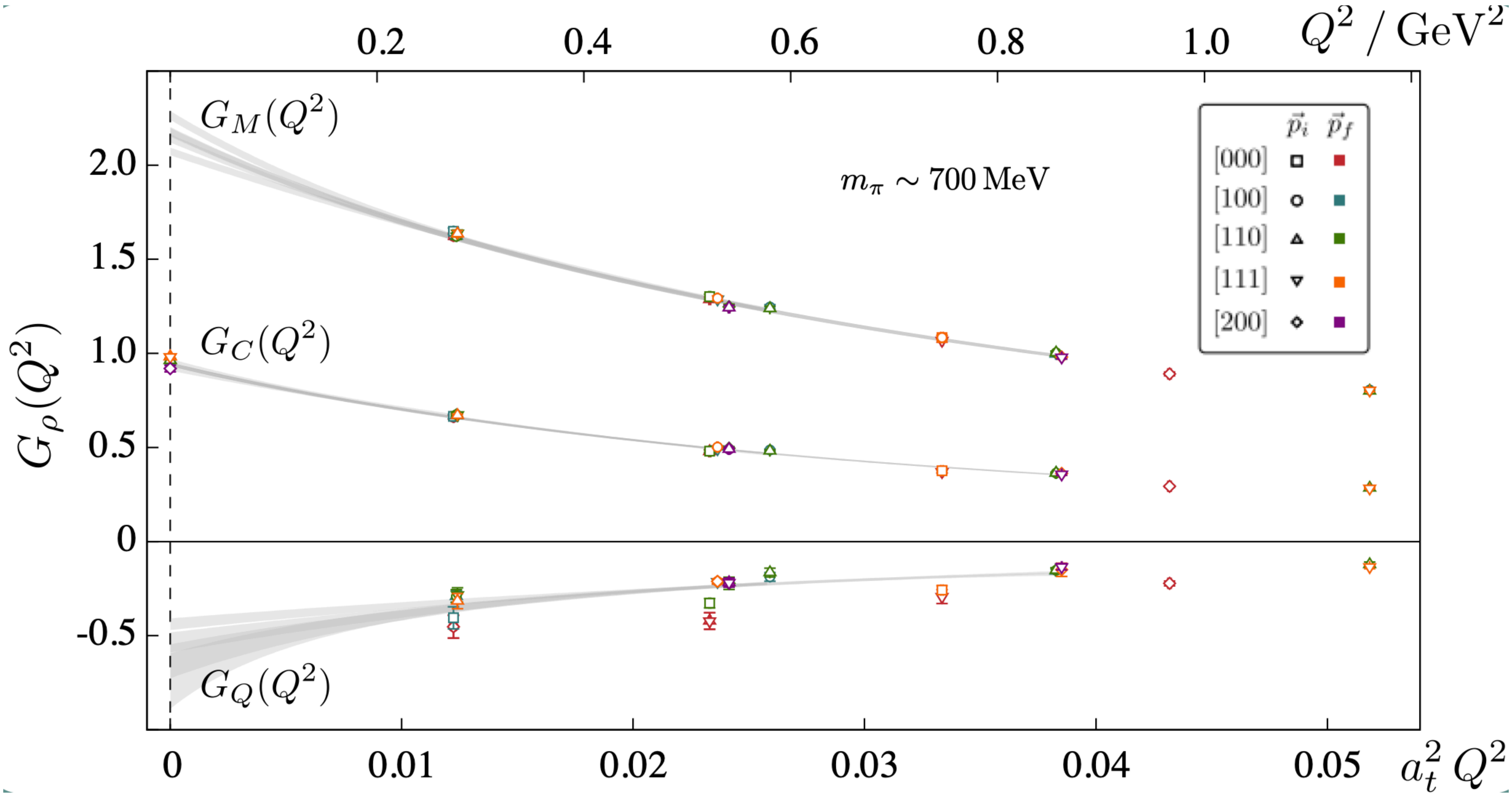
Form-factor of a stable resonance

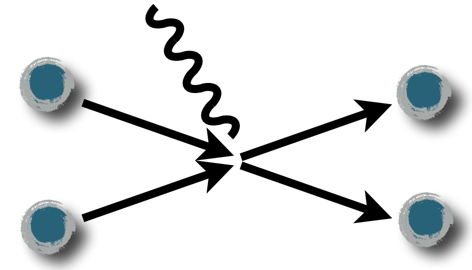


• Shultz, Dudek, and Edwards (2015) •

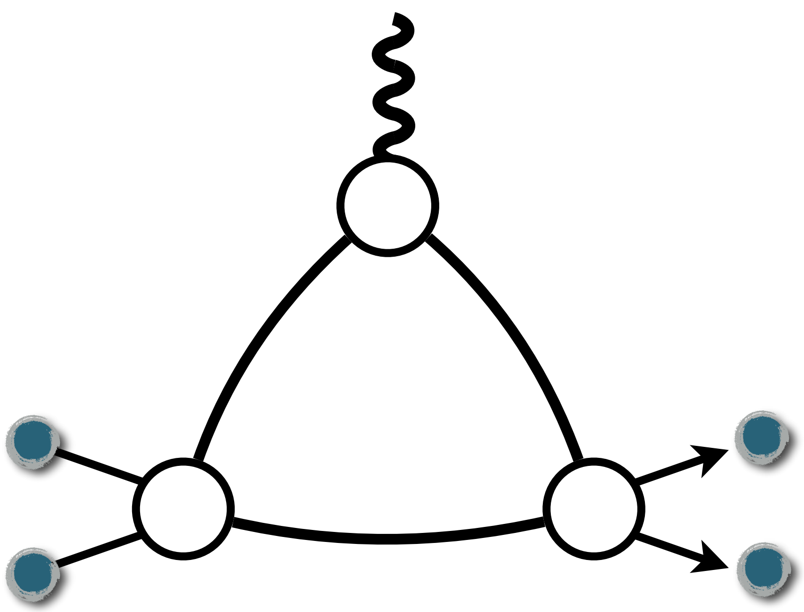
$2 + \mathscr{J} \to 2$

- Fully developed formalism for multi-hadron form factors

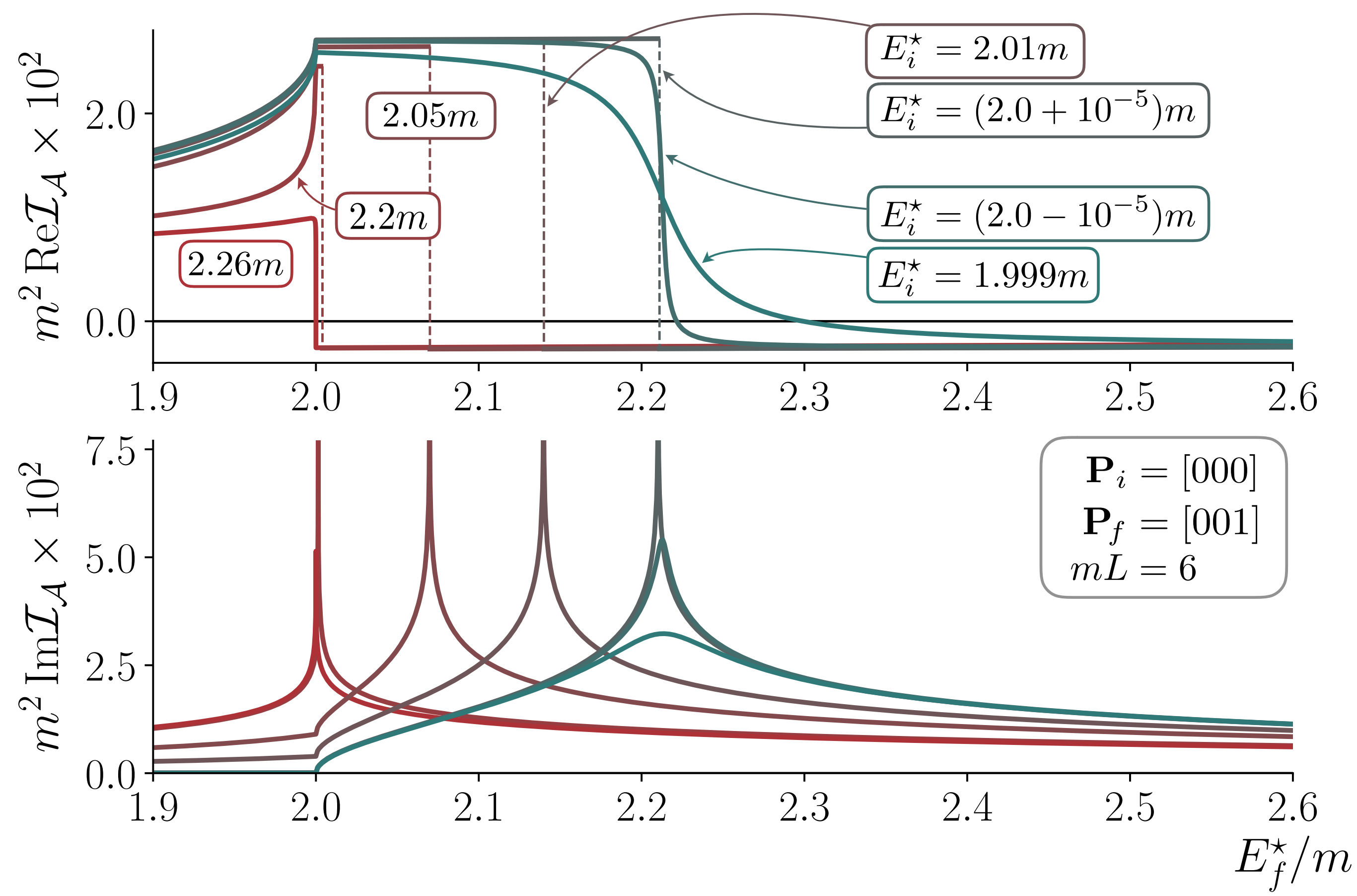
$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi\pi, \text{in} \rangle \equiv$$



- Continuation to the pole → ***resonance form factors***
- Must carefully treat *triangle* singularities



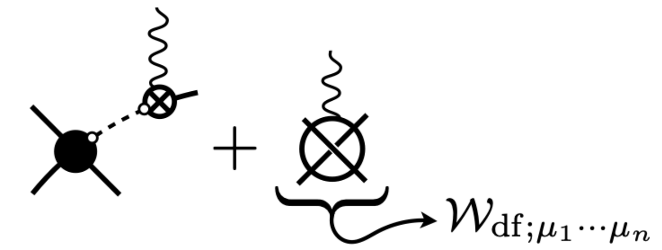
$\sim$



Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

Technical summary of the result

$$\mathcal{W}_{\mu_1\cdots\mu_n} =$$



- ☐ Define the amplitude

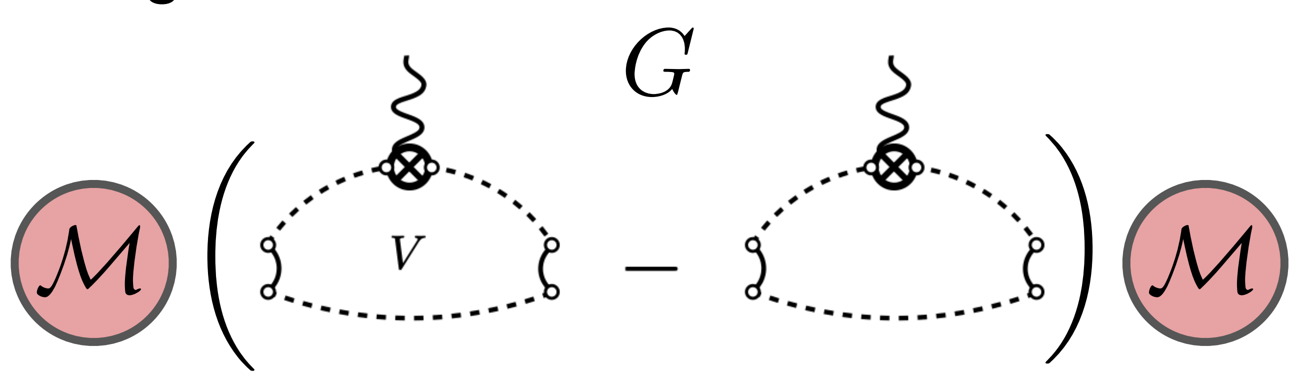
$$\mathcal{W}_{\mu_1\cdots\mu_n}(P_f,k';P_i,k) \equiv \langle P_f,k';\mathrm{out}|\mathcal{J}_{\mu_1\cdots\mu_n}(0)|P_i,k;\mathrm{in}\rangle_{\mathrm{conn.}}\,.$$

- ☐ Define a pole-free version

$$\mathcal{W}_{\mathrm{df};\mu_1\cdots\mu_n} \equiv \mathcal{W}_{\mu_1\cdots\mu_n} - i\overline{\mathcal{M}}(P_f,k',k)\frac{i}{(P_f-k)^2-m_1^2}w_{\mu_1\cdots\mu_n}$$

- ☐ Define a finite-volume version with the triangle corrected

$$\mathcal{W}_{L,\mathrm{df}}^{\mu_1\cdots\mu_n}(P_f,P_i,L) - \mathcal{W}_{\mathrm{df}}^{\mu_1\cdots\mu_n}(s_f,s_i,Q^2) \equiv$$

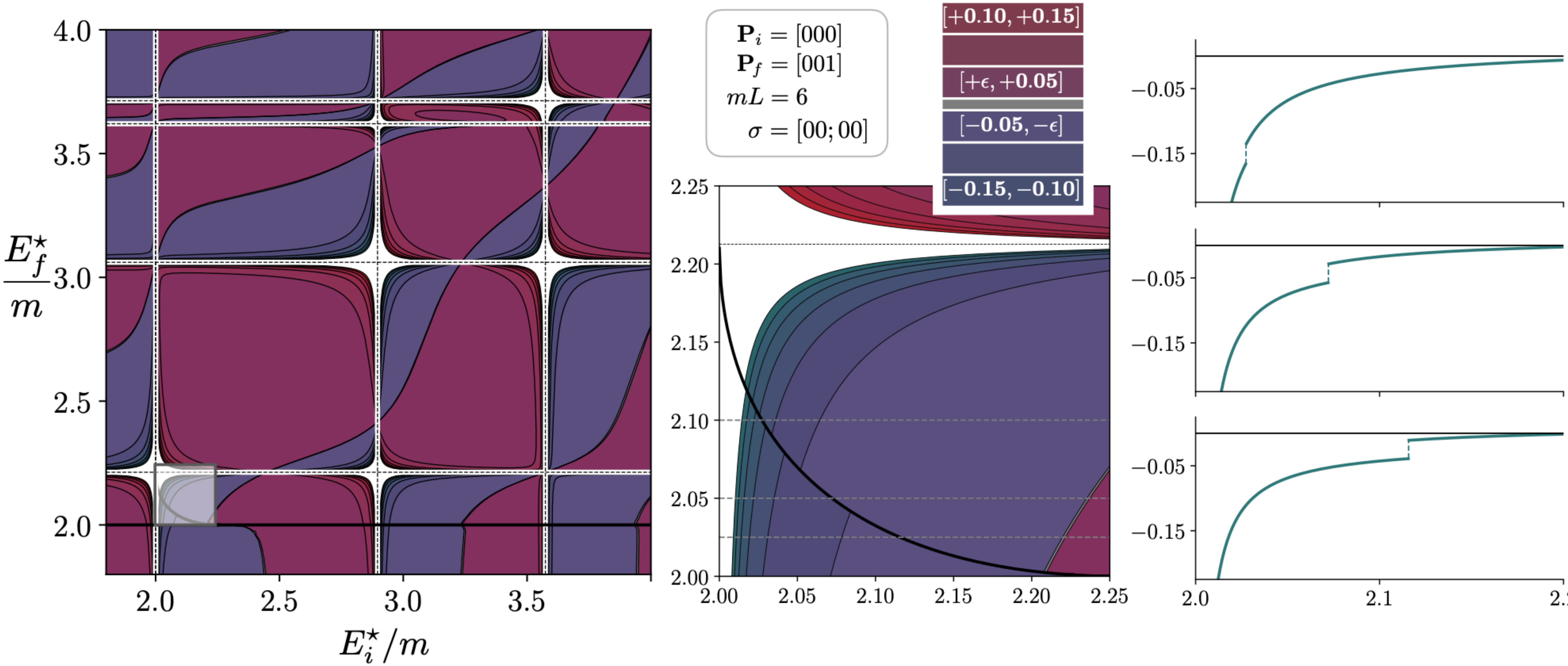


- ☐ Relate this to a finite-volume matrix element

$$\left|\langle P_f,L|\mathcal{J}^{\mu_1\cdots\mu_n}(0)|P_i,L\rangle\right|^2 = \frac{1}{L^6}\mathrm{Tr}\left[\mathcal{R}(P_i)\mathcal{W}_{L,\mathrm{df}}^{\mu_1\cdots\mu_n}(P_i,P_f,L)\mathcal{R}(P_f)\mathcal{W}_{L,\mathrm{df}}^{\mu_1\cdots\mu_n}(P_f,P_i,L)\right]$$

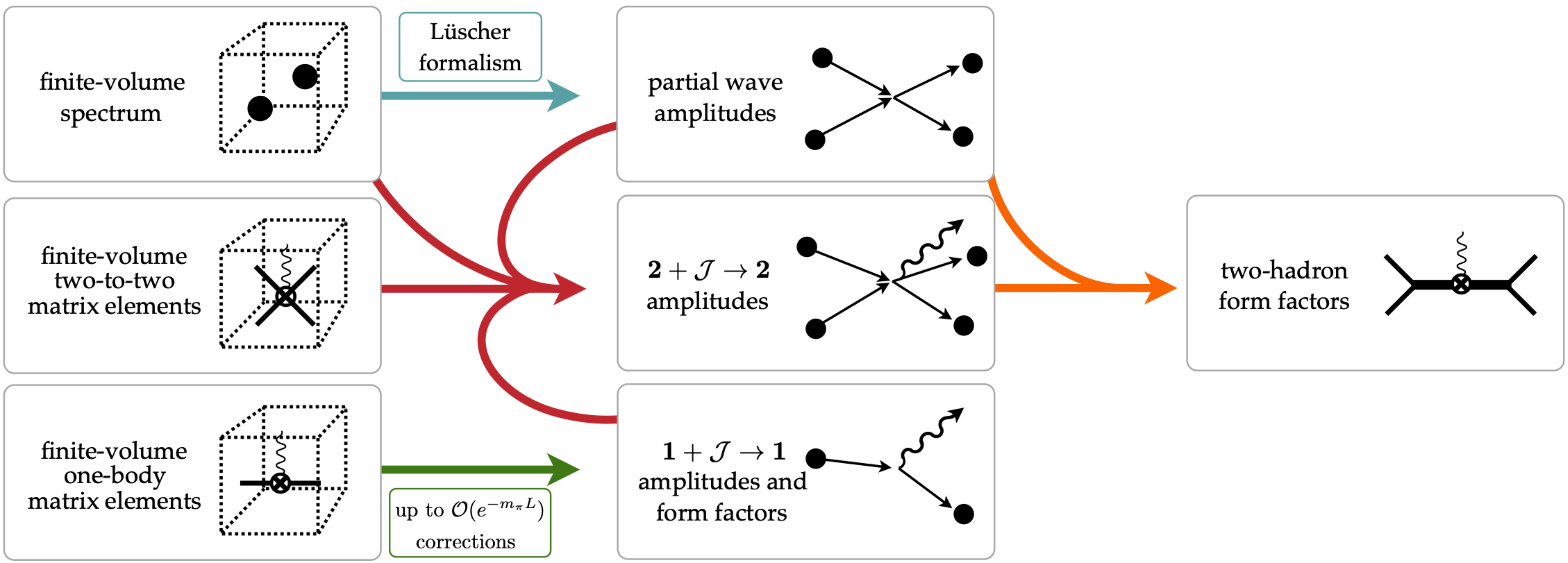
Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

Visualizing G



Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

More complicated work flow



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- ☐ Watch Raul's talk at lattice 2021!

Four-point functions

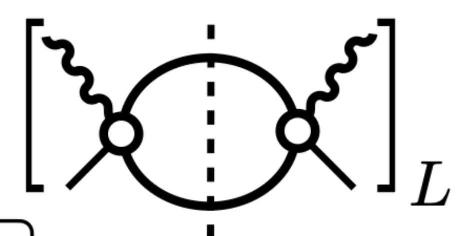
Physical long-range processes: $\mathcal{T} \sim \int_{-\infty}^{\infty} dt\, e^{iq_0 t} \langle n_f |\, \mathcal{J}_{2,M}(t)\, \mathcal{J}_1(0)\, | n_i \rangle_\infty$

Naïve guess: $\mathcal{T} \sim \int_{-\infty}^{\infty} d\tau\, e^{q_0 \tau} \langle n_f |\, \mathcal{J}_{2,E}(\tau)\, \mathcal{J}_1(0)\, | n_i \rangle_L$

Hidden dragons [ ]:

- ☐ The integral doesn't always converge!
- ☐ The divergences are volume dependent!

$$\int_{-\infty}^{\infty} d\tau\, e^{q_0 \tau} \langle n_f |\, \mathcal{J}_{2,E}(\tau)\, \mathcal{J}_1(0)\, | n_i \rangle_L \sim \int_{-\infty}^{\infty} d\tau\, e^{(q_0 + E_f - E_n)\tau} \langle n_f |\, \mathcal{J}_{2,E}(0)\, | n \rangle_L \xrightarrow{q_0 + E_f = E_n} \infty$$



associated with intermediate states going on-shell

Christ et al. (2015) • Briceno, Davoudi, MTH, Schindler, Baroni (2019)

https://indico.cern.ch/event/1006302/contributions/4373256/

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