UK Research and Innovation

QCD (and QED) in a Euclidean finite volume

EuroPLEx Summer School 2021

Maxwell T. Hansen

August 23rd-27th

THE UNIVERSITY *of* EDINBURGH

Five-day outline

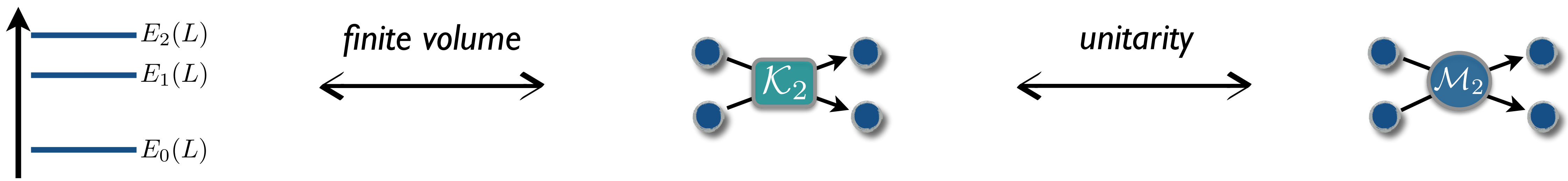
- [x] **Warm-up and definitions**
  - [x] Meaning of Euclidean
  - [x] Finite-volume set-up
- [x] **e^{-mL} volume effects**
  - [x] Mass, matrix element in $\lambda\phi^4, g\phi^3$
  - [x] Pion mass
  - [x] LO-HVP for $(g-2)_\mu$
  - [x] Bethe-Salpeter kernel
- [x] **$2 \to 2$ finite-volume (f.v.) formalism**
  - [x] Scattering basics
  - [x] Derivation
  - [x] Generalizations
  - [x] Applications
- [x] **$(1+)\mathcal{J} \to 2$ f.v. formalism**
  - [x] Derivation
  - [x] Expansions
  - [x] Applications
- [x] **$2 + \mathcal{J} \to 2$ f.v. formalism**
  - [x] Derivation
  - [x] Testing the result
  - [x] Numerical explorations
- [x] **Non-local matrix elements**
  - [x] Derivation
  - [x] Applications
- [ ] **$3 \to 3$ f.v. formalism**
  - [ ] New complications
  - [ ] Derivation ($E_n(L)$ to $\mathcal{K}_{\mathrm{df},3}$)
  - [ ] Integral equations ($\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$)
  - [ ] Testing the result
  - [ ] Numerical explorations/calculations
- [ ] **Finite-volume QED + QCD**
  - [ ] Different formalisms
  - [ ] Basic examples

Five-day outline

- [x] Warm-up and definitions
  - [x] Meaning of Euclidean
  - [x] Finite-volume set-up
- [x] e^{-mL} volume effects
  - [x] Mass, matrix element in $\lambda\phi^4, g\phi^3$
  - [x] Pion mass
  - [x] LO-HVP for $(g-2)_\mu$
  - [x] Bethe-Salpeter kernel
- [x] $2 \to 2$ finite-volume (f.v.) formalism
  - [x] Scattering basics
  - [x] Derivation
  - [x] Generalizations
  - [x] Applications
- [x] $(1+)\mathscr{J} \to 2$ f.v. formalism
  - [x] Derivation
  - [x] Expansions
  - [x] Applications
- [x] $2 + \mathscr{J} \to 2$ f.v. formalism
  - [x] Derivation
  - [x] Testing the result
  - [x] Numerical explorations
- [x] Non-local matrix elements
  - [x] Derivation
  - [x] Applications
- [ ] $3 \to 3$ f.v. formalism
  - [ ] New complications
  - [ ] Derivation ($E_n(L)$ to $\mathscr{K}_{\text{df},3}$)
  - [ ] Integral equations ($\mathscr{K}_{\text{df},3}$ to $\mathscr{M}_3$)
  - [ ] Testing the result
  - [ ] Numerical explorations/calculations
- [ ] Finite-volume QED + QCD
  - [ ] Different formalisms
  - [ ] Basic examples

Result

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0 \qquad F(P,L) \equiv \text{Matrix of known geometric functions}$$



Holds only for two-particle energies $s < (4m)^2$

Neglects e^{-mL}

Generalized to ***non-degenerate masses, multiple channels, spinning particles***

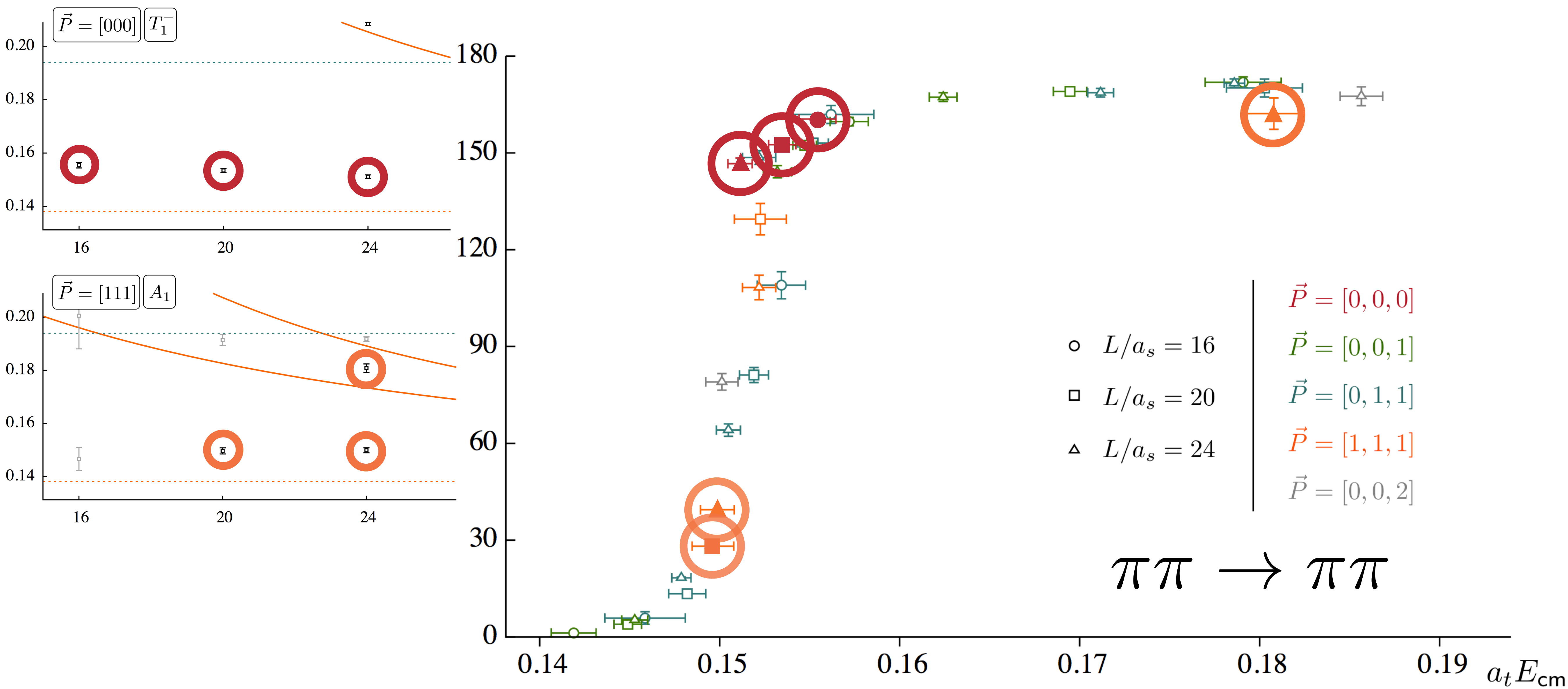
Encodes angular momentum mixing

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
Li, Liu (2013) • Briceño (2014)

Using the result

- Single-channel case *(pions in a p-wave)*

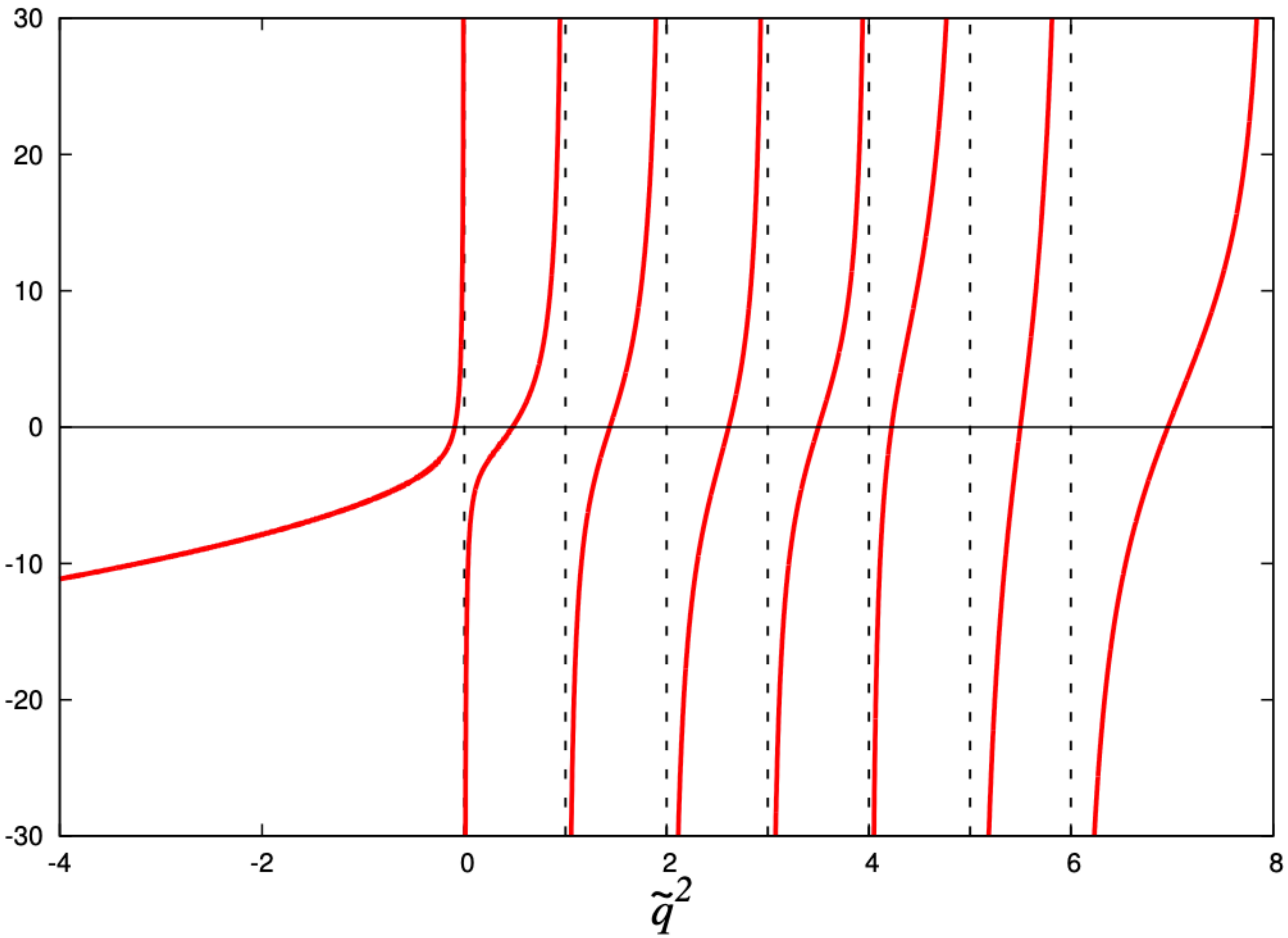
$$\mathcal{K}(s_n)^{-1} = \rho \cot\delta(s_n) = -F(E_n, \vec{P}, L)$$



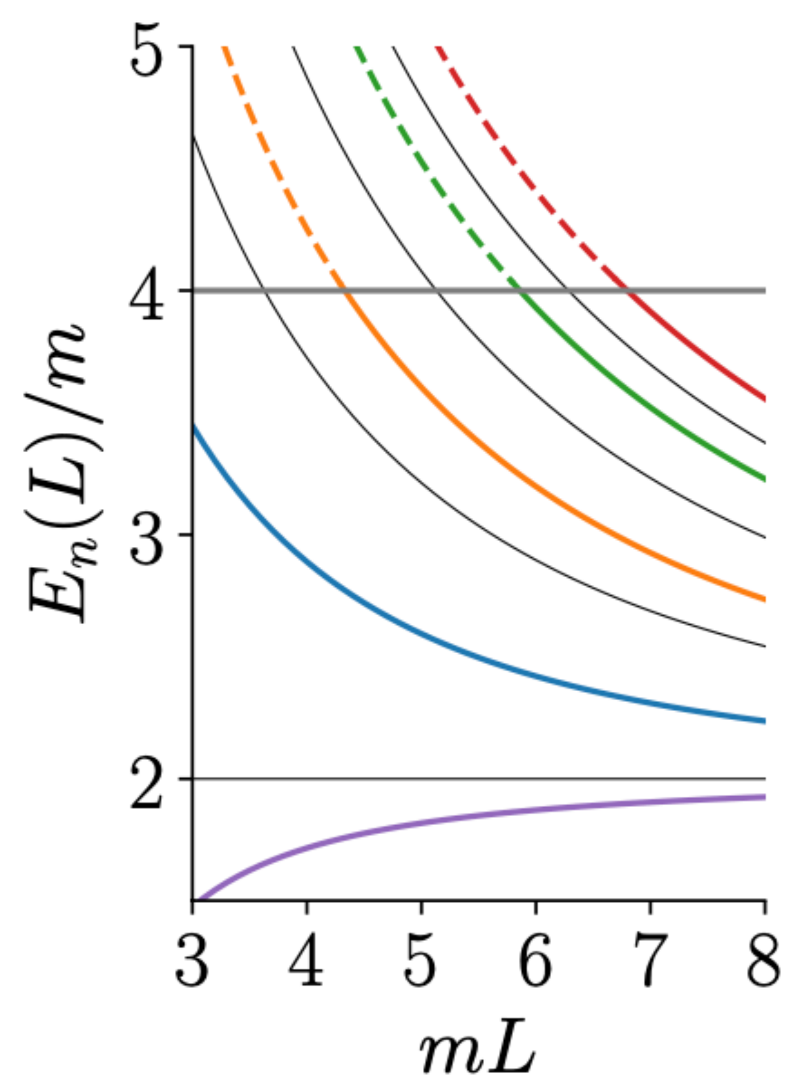
• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

Visualizing the result

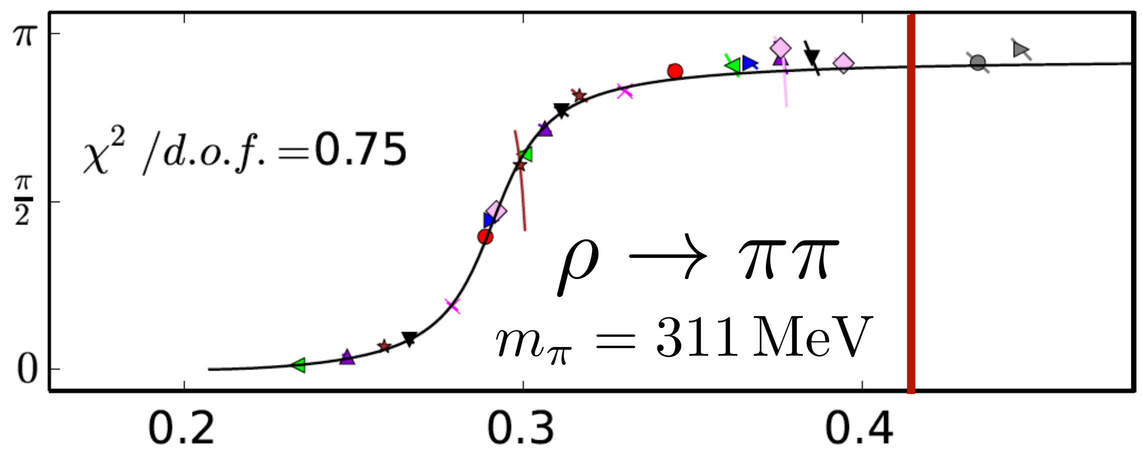
$$p \cot \delta(p) = \frac{1}{\pi L}\left[\sum_{\boldsymbol{n}} - \text{p.v.} \int d^3\boldsymbol{n}\right] \frac{1}{\boldsymbol{n}^2 - q^2}$$



- plot taken from Luu and Savage (2011) •



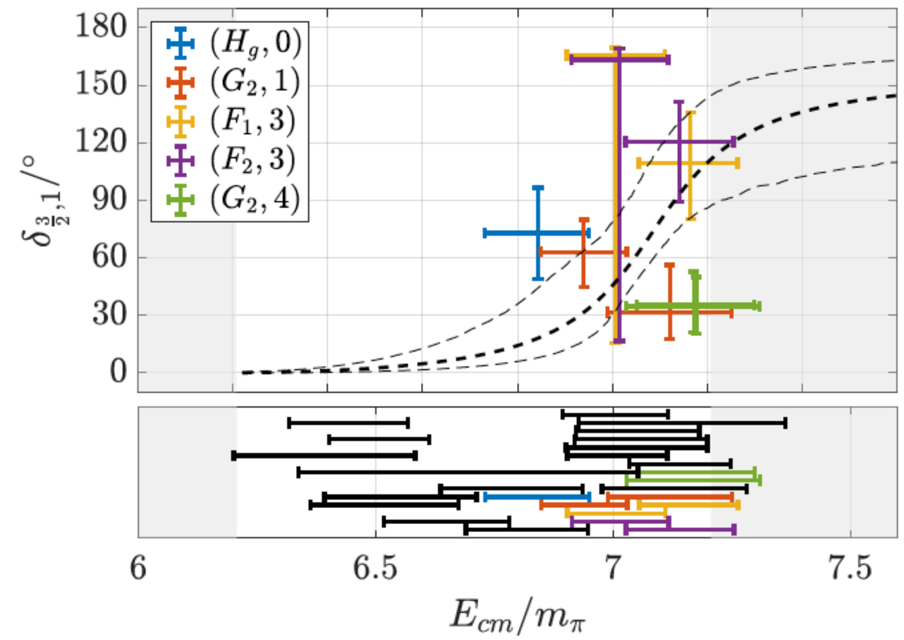
Many applications



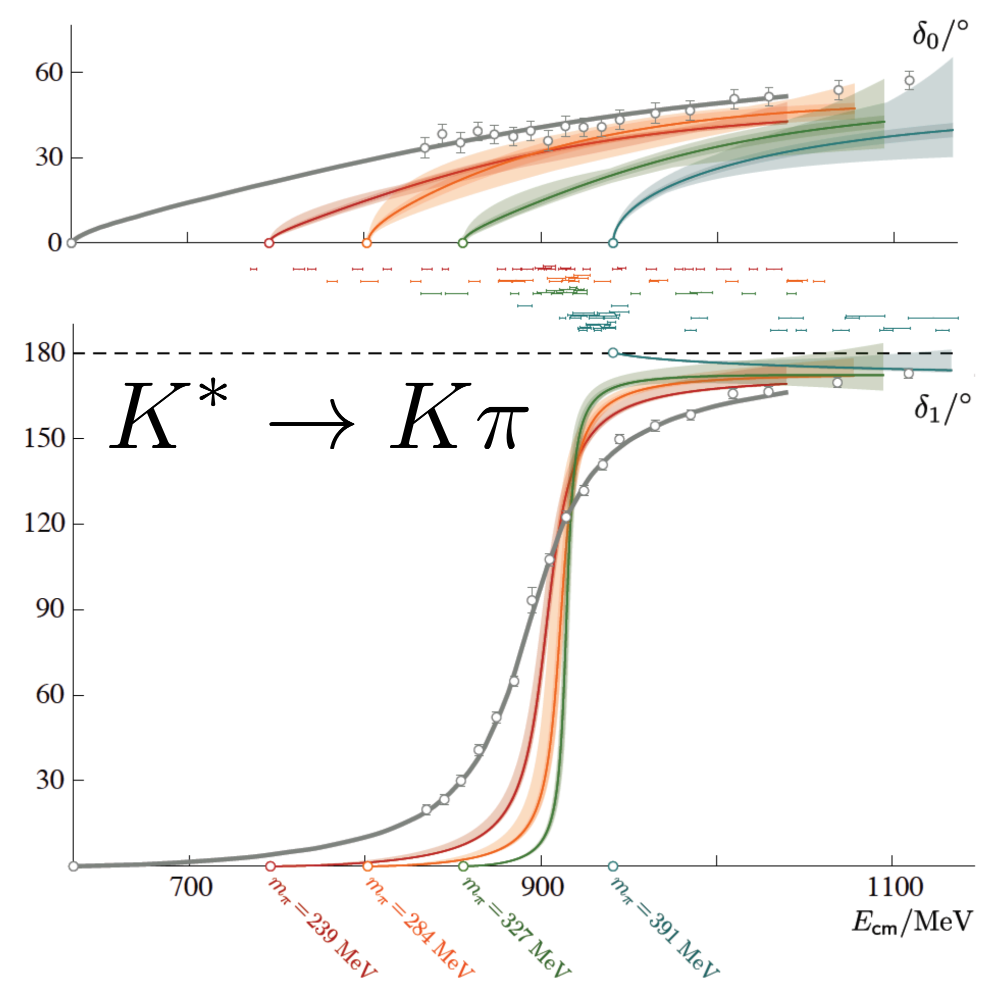
• Erben et al. (2019) •

$\Delta \to N\pi$

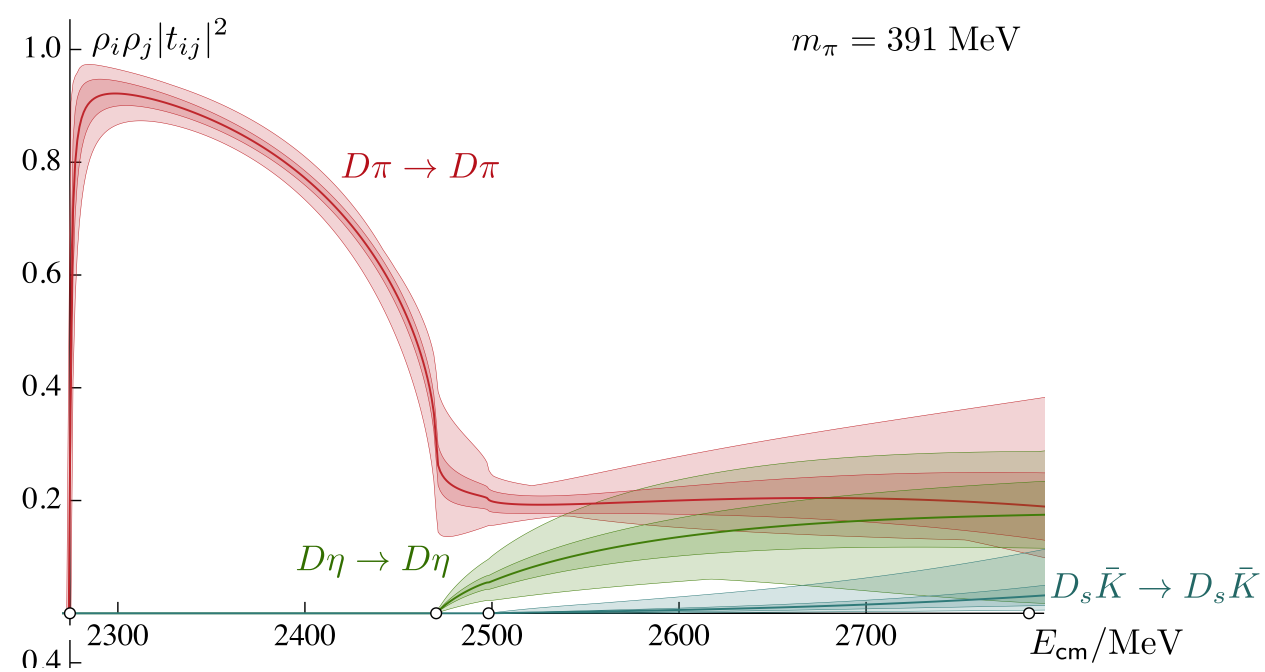
$m_\pi = 200$ MeV



• Andersen et al. (2019) •

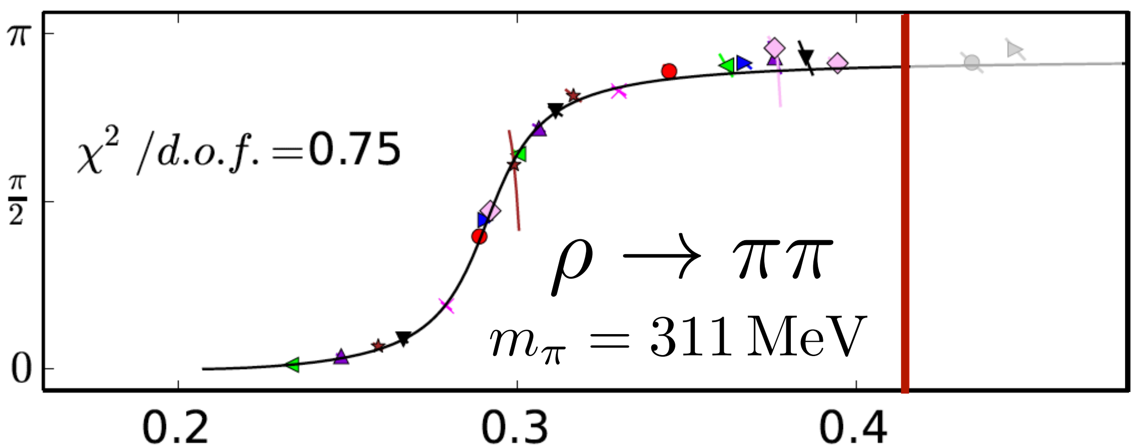


• Wilson et al. *PRL123* (2019) •



• Moir et al. *JHEP 1610* (2016) •

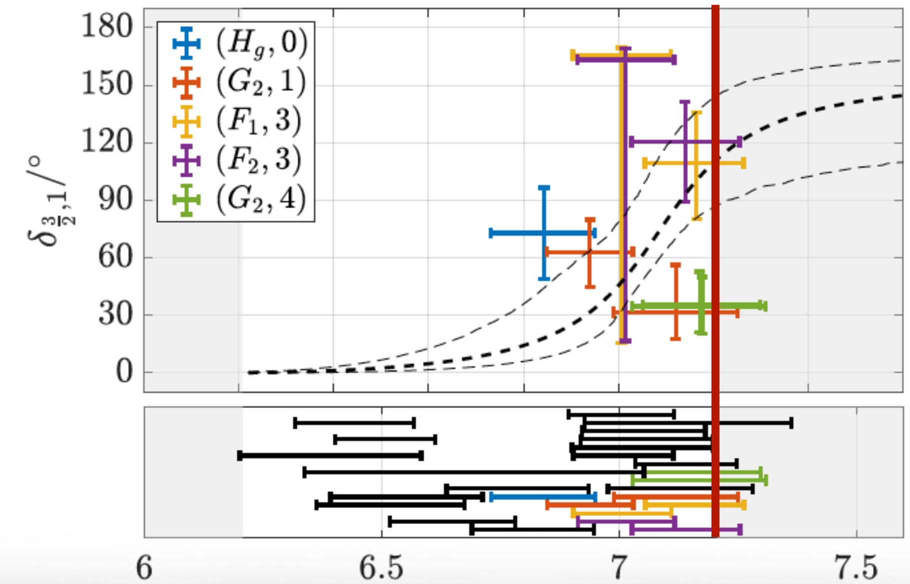
Many applications



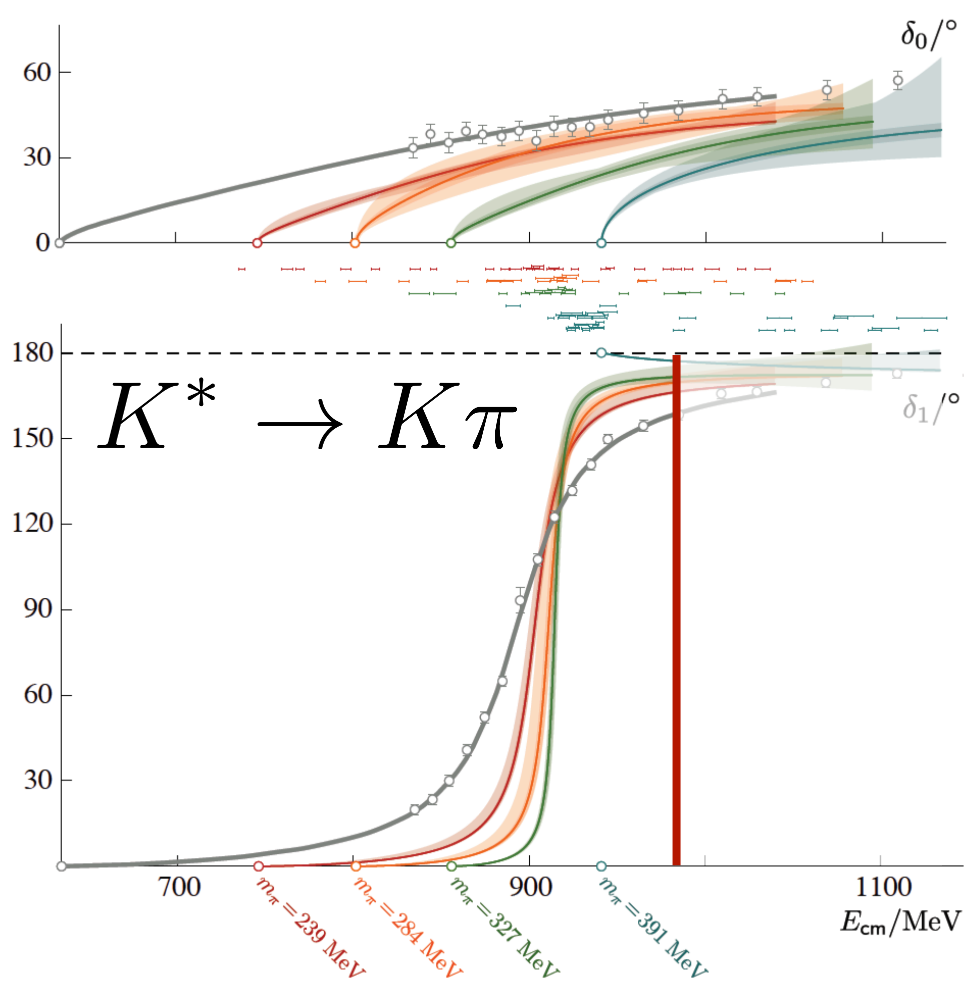
• Erben et al. (2019) •

$\Delta \to N\pi$

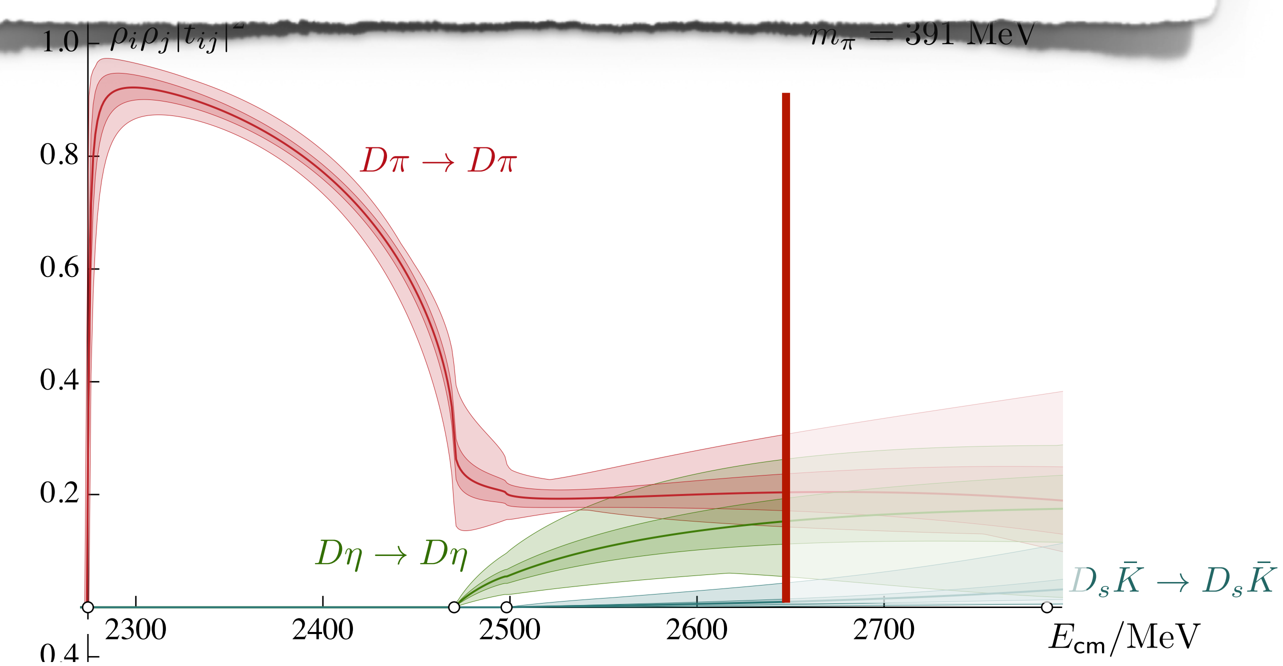
$m_\pi = 200$ MeV



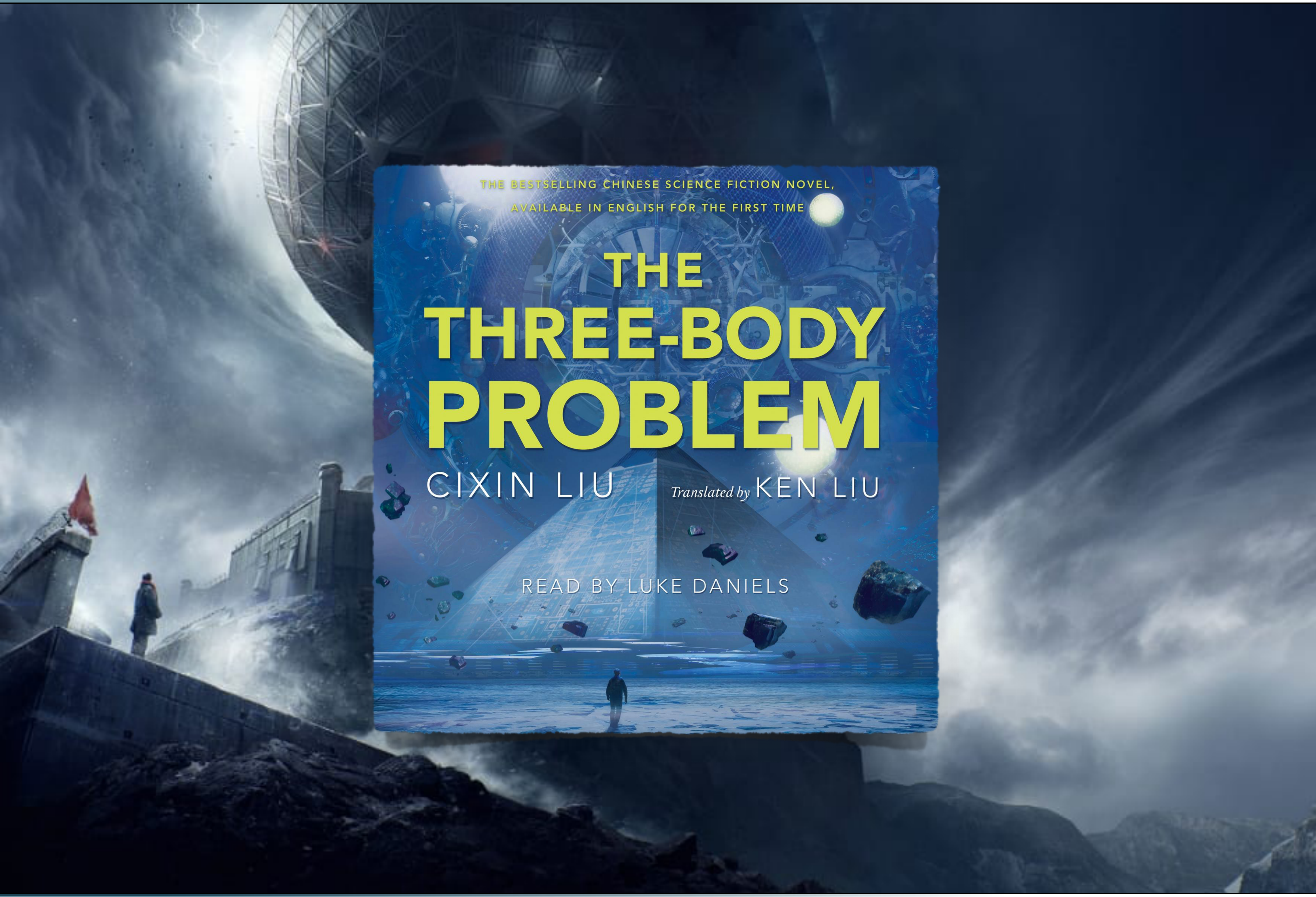
Formalism only strictly holds up to lowest 3- or 4-particle threshold



• Wilson et al. *PRL123* (2019) •



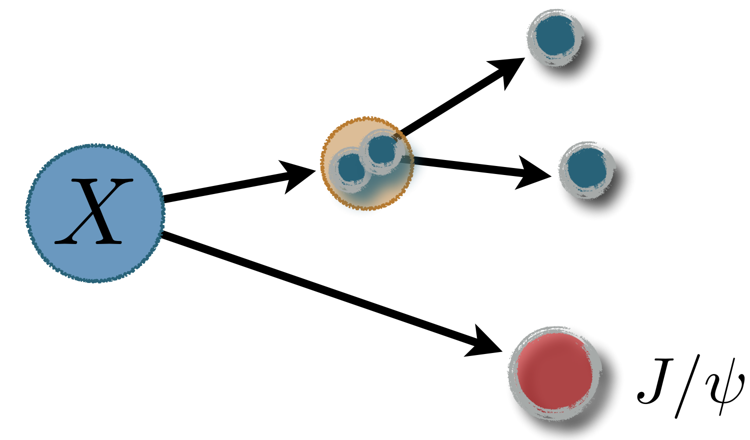
• Moir et al. *JHEP 1610* (2016) •


THE BESTSELLING CHINESE SCIENCE FICTION NOVEL,
AVAILABLE IN ENGLISH FOR THE FIRST TIME
THE THREE-BODY PROBLEM
CIXIN LIU
Translated by KEN LIU
READ BY LUKE DANIELS

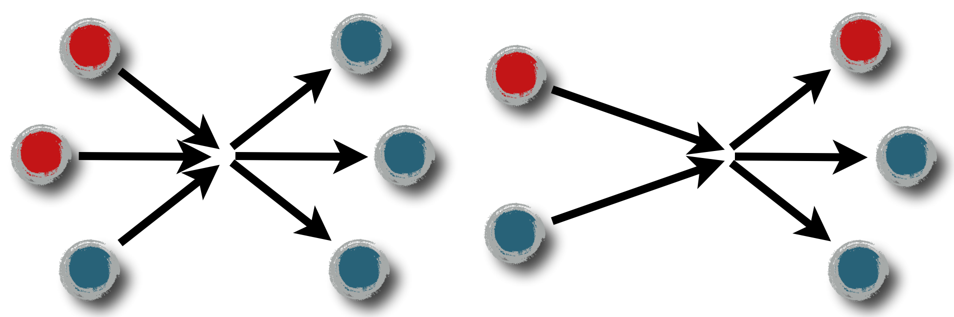
3-particle amplitudes

2-to-2 only samples J^P 0^+ 1^- $2^+ \cdots$

many interesting resonances have significant 3-body decays



*Goal: **finite-volume + unitarity formalism*** for generic two- and three-particle systems



Applications…

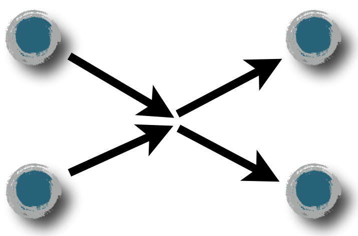
exotic resonance pole positions, couplings, quantum numbers

$\omega(782),\, a_1(1420) \to \pi\pi\pi$ $\quad X(3872) \to J/\psi\pi\pi$ $\quad X(3915)[Y(3940)] \to J/\psi\pi\pi$

form factors and transitions

and much more!… *(3-body forces, weak transitions, gluons content)*

Complication: degrees of freedom

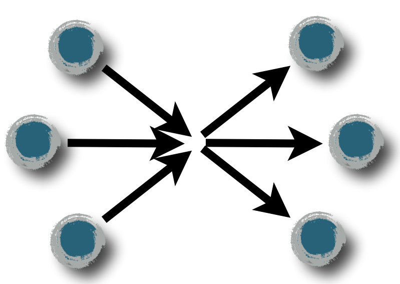


12 momentum components

-10 Poincaré generators

2 degrees of freedom

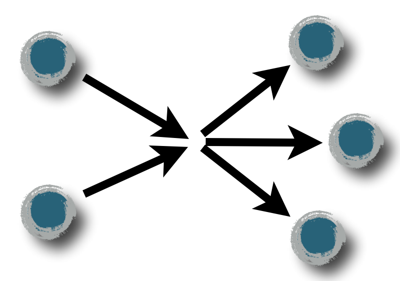
$$\vec{p}_1 + \vec{p}_2 \rightarrow \vec{p}_3 + \vec{p}_4 \longrightarrow \text{Mandelstam } s, t$$



18 momentum components

-10 Poincaré generators

8 degrees of freedom



15 momentum components

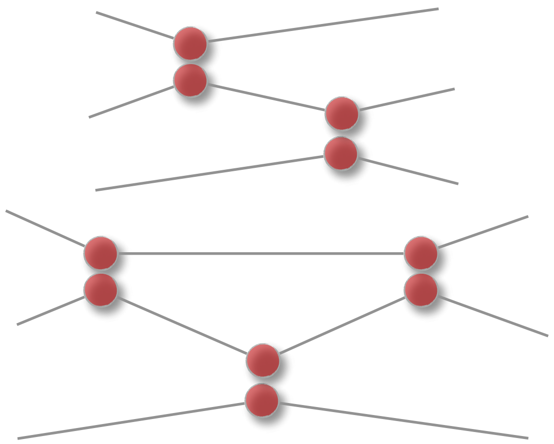
-10 Poincaré generators

5 degrees of freedom

Complication: on-shell states

- Classical pairwise scattering

for $m_1 = m_2 = m_3$ up to ***3*** binary collisions are possible



Dispersion Relations for Three-Particle Scattering Amplitudes. I*

MORTON RUBIN
Physics Department, University of Wisconsin, Madison, Wisconsin

AND

ROBERT SUGAR
Physics Department, Columbia University, New York, New York

AND

GEORGE TIKTOPOULOS
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received 31 January 1966)

$$b=\frac{(m_1+m_3)(m_2+m_3)}{m_1 m_2}$$

It follows that if

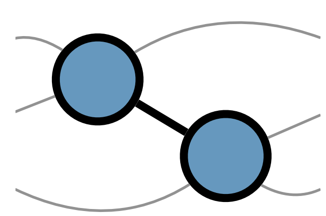
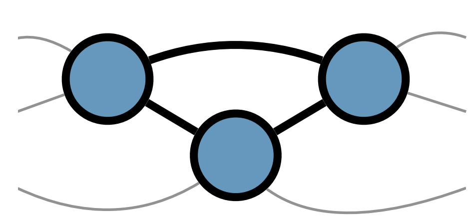
$$b^{n-2}(b-1)>1\,, \qquad \text{(IV.18)}$$

then $2n+1$ successive binary collisions are kinematically impossible.

$m_1 = m_2 = m_3 - \varepsilon$:
4 collisions possible
$\pi\pi K$

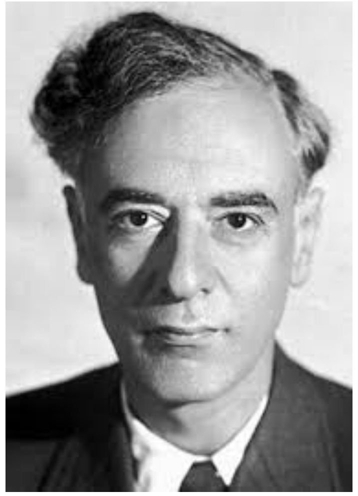
$b < 2$
5 collisions possible
$\pi K K$

- Correspond to Landau singularities

$$i\mathcal{M}_{3\to 3} \equiv \text{fully connected correlator} = $$  $+$  $+ \cdots$

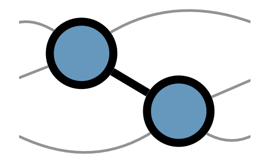
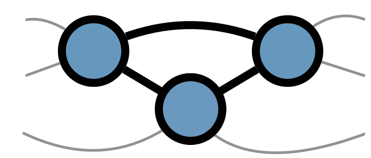
complicate analyticity & unitarity

difficult to disentangle kinematic singularities from resonance poles



Two key observations

- ☐ Intermediate $K_{df,3}$ removes singularities

$i\mathcal{K}_{\mathrm{df},3} \equiv$ fully connected diagrams w/ ***PV pole prescription*** $-$  $+$  $+ \cdots$

same degrees of freedom as M_3 smooth real function   relation to M_3 = known

- ☐ $K_{df,3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\mathrm{df},3}(p_3, p_2, p_1; k_3, k_2, k_1) = \mathcal{K}^{\mathrm{iso},0}_{\mathrm{df},3} + \mathcal{K}^{\mathrm{iso},1}_{\mathrm{df},3}\,\Delta + \cdots \qquad \Delta = \frac{s-(3m)^2}{(3m)^2}$$

smooth real function

analogous to effective range expansion

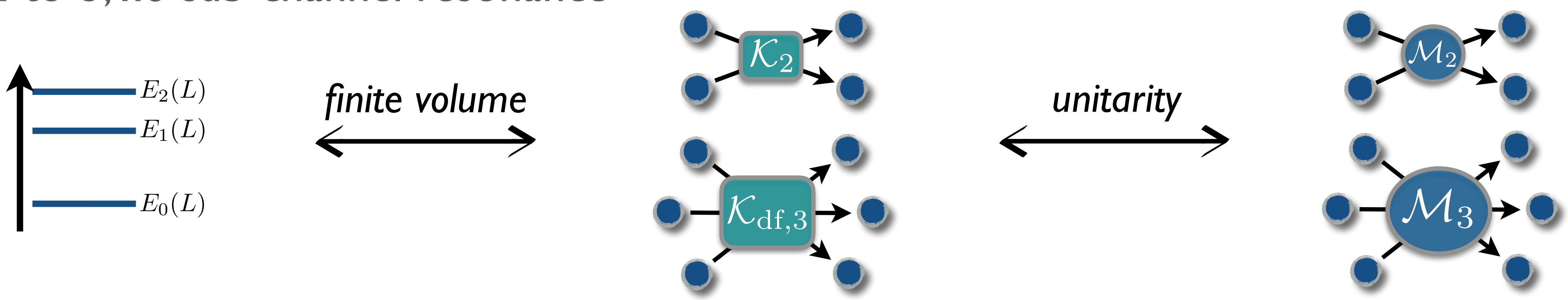
$$p\cot\delta = -\frac{1}{a} + \frac{1}{2}\,r\,p^2 + \mathcal{O}(p^4)$$

gives handle on many *DOFs*

Status…

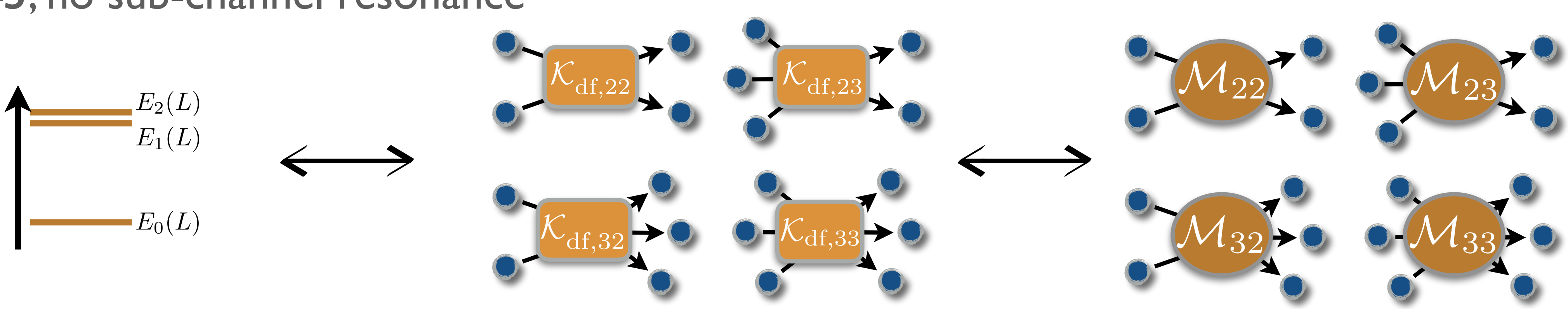
- [ ] General relation between *energies* and *two-and-three scalar scattering*

No **2-to-3,** ***no*** **sub-channel resonance**



• MTH, Sharpe (2014, 2015) •

2-to-3, no sub-channel resonance



• Briceño, MTH, Sharpe (2017) •

Including sub-channel resonances + *different isospins*

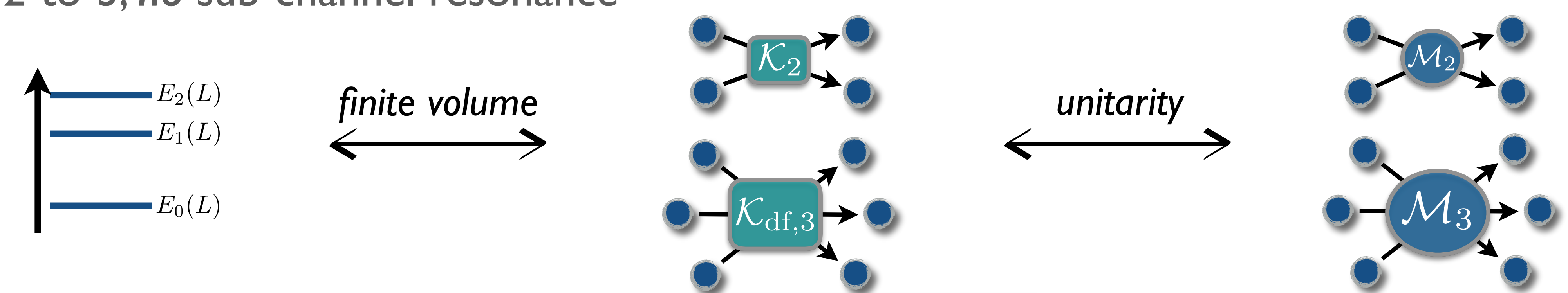
$$\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$$

• Briceño, MTH, Sharpe (2018) • Blanton, Briceño, MTH, Romero-López, Sharpe (2019 + *in progress*) •

Status…

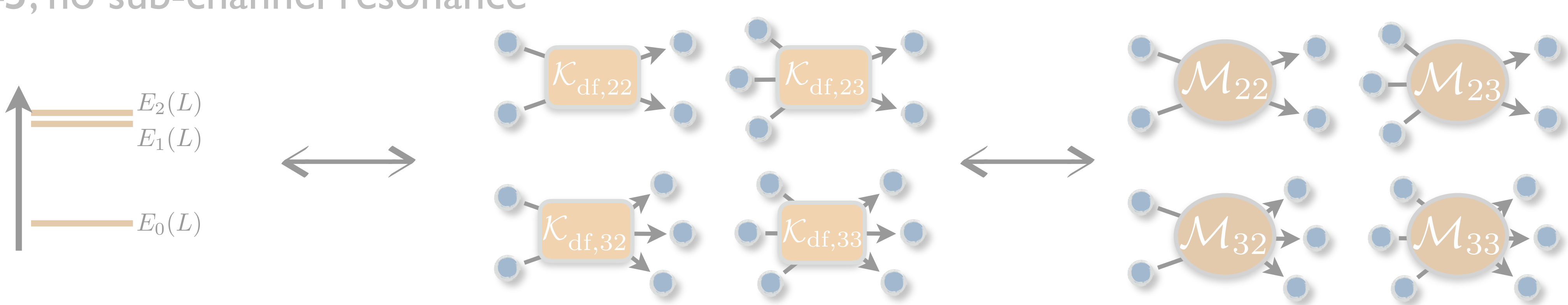
- [ ] General relation between *energies* and *two-and-three scalar scattering*

No **2-to-3,** ***no*** **sub-channel resonance**



• MTH, Sharpe (2014, 2015) •

2-to-3**, no sub-channel resonance**



• Briceño, MTH, Sharpe (2017) •

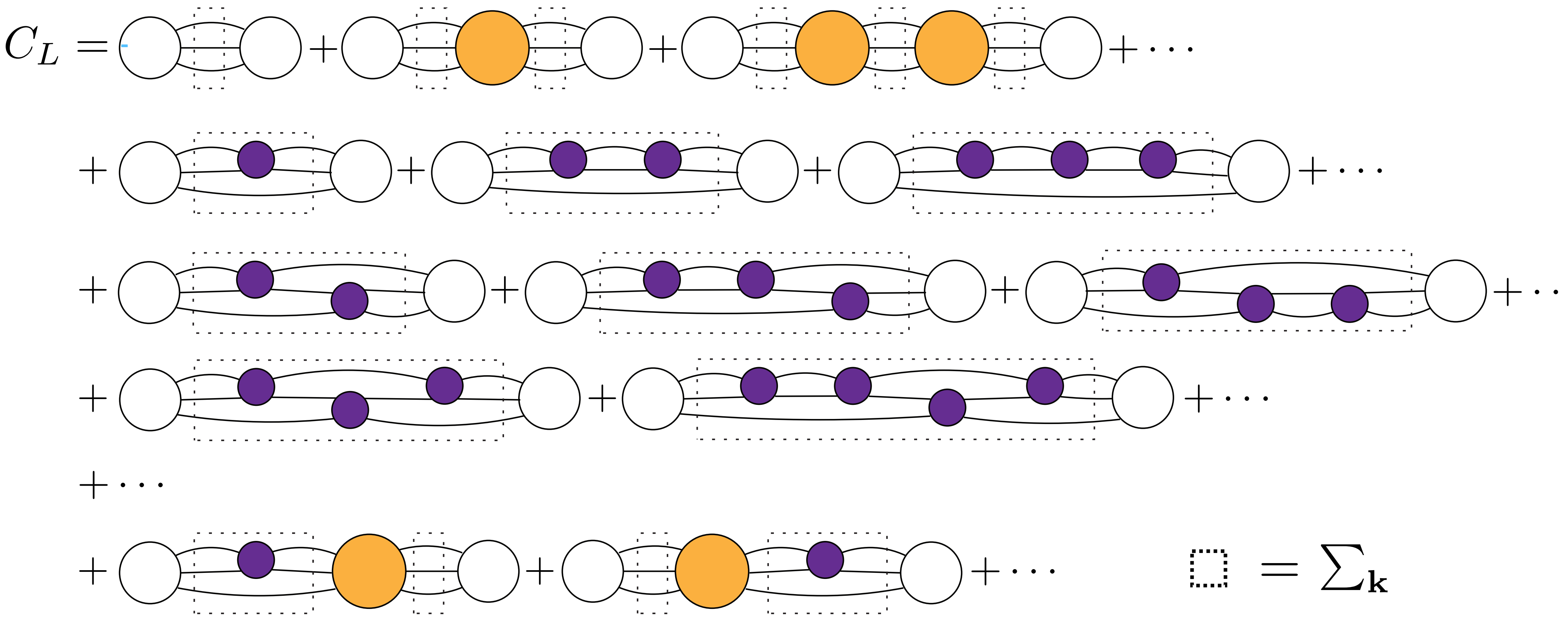
Including sub-channel resonances + *different isospins*

$$\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$$

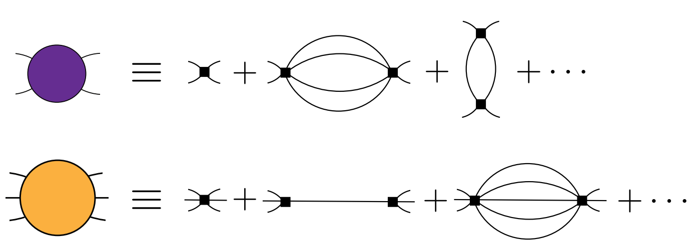
• Briceño, MTH, Sharpe (2018) • Blanton, Briceño, MTH, Romero-López, Sharpe (2019 + *in progress*) •

Derivation

- Study 3-body correlator in an ***all-orders skeleton expansion***



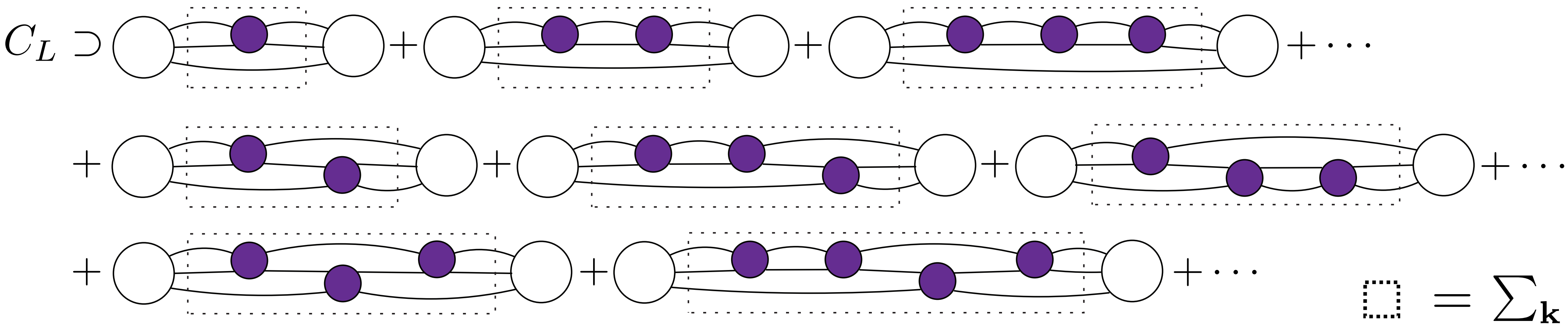
Here… assume no 2-to-3 allowed



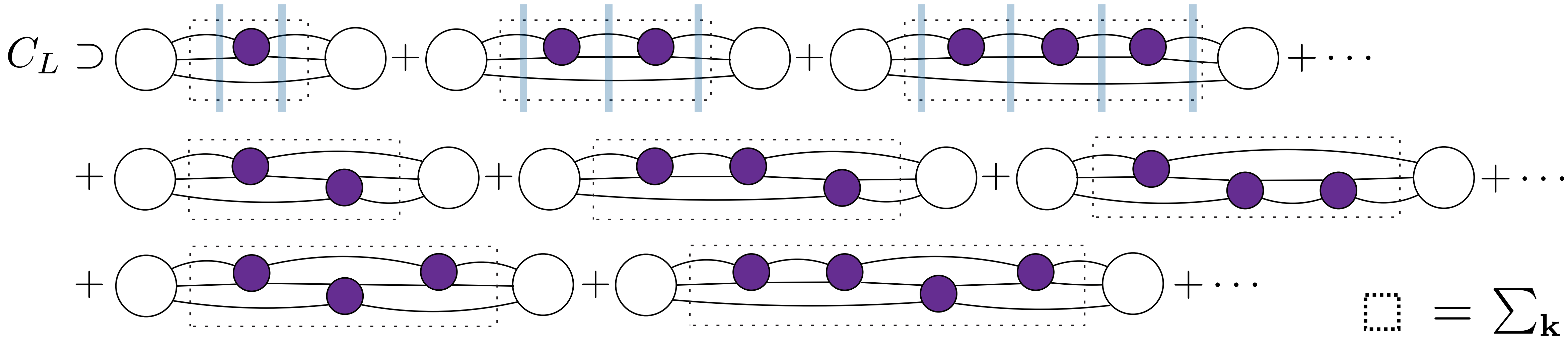
kernels have suppressed L dependence

lines = fully dressed hadrons

Two types of cuts

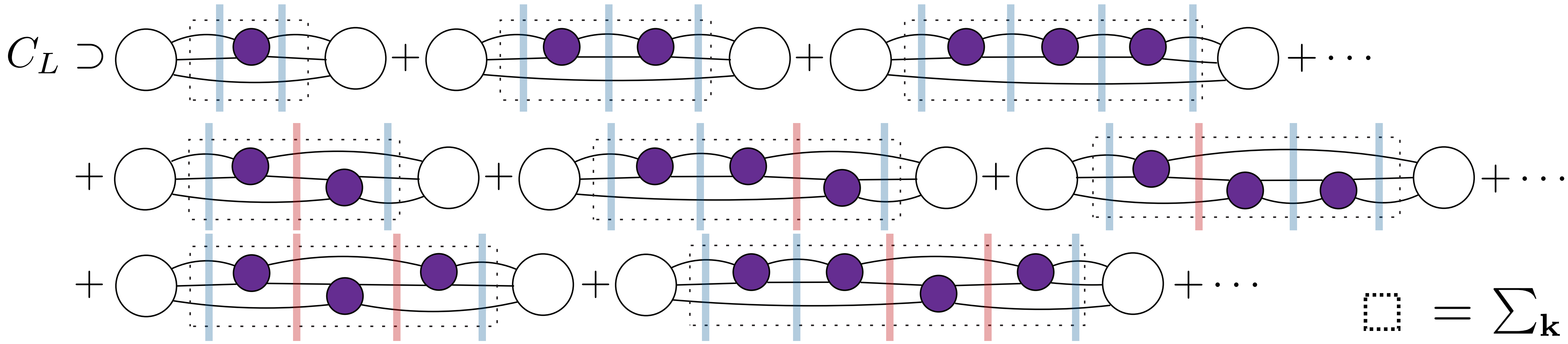


Two types of cuts



$$\mathbf{A}_3' \, \mathbf{F} \, \mathbf{K}_2 \, \mathbf{F} \, \mathbf{A}_3 + \mathbf{A}_3' \, \mathbf{F} \, [\mathbf{K}_2 \, \mathbf{F}]^2 \, \mathbf{A}_3 + \mathbf{A}_3' \, \mathbf{F} \, [\mathbf{K}_2 \, \mathbf{F}]^3 \, \mathbf{A}_3 + \cdots = \mathbf{A}_3' \, \mathbf{F} \frac{1}{1 - \mathbf{K}_2 \mathbf{F}} \mathbf{K}_2 \mathbf{F} \, \mathbf{A}_3$$

Two types of cuts



$$\mathbf{A}_3' \mathbf{F} \mathbf{K}_2 \mathbf{F} \mathbf{A}_3 + \mathbf{A}_3' \mathbf{F} [\mathbf{K}_2 \mathbf{F}]^2 \mathbf{A}_3 + \mathbf{A}_3' \mathbf{F} [\mathbf{K}_2 \mathbf{F}]^3 \mathbf{A}_3 + \cdots = \mathbf{A}_3' \mathbf{F} \frac{1}{1 - \mathbf{K}_2 \mathbf{F}} \mathbf{K}_2 \mathbf{F} \mathbf{A}_3$$

$$\mathbf{A}_3' \mathbf{F} \mathbf{K}_2 \mathbf{G} \mathbf{K}_2 \mathbf{F} \mathbf{A}_3 + \cdots$$

$$C_L - C_\infty \supset \mathbf{A}_3' \mathbf{F}_{33} \mathbf{A}_3 \qquad \mathbf{F}_{33} \equiv \frac{1}{3} \mathbf{F} + \mathbf{F} \mathbf{K}_2 \frac{1}{1 - (\mathbf{F} + \mathbf{G}) \mathbf{K}_2} \mathbf{F}$$

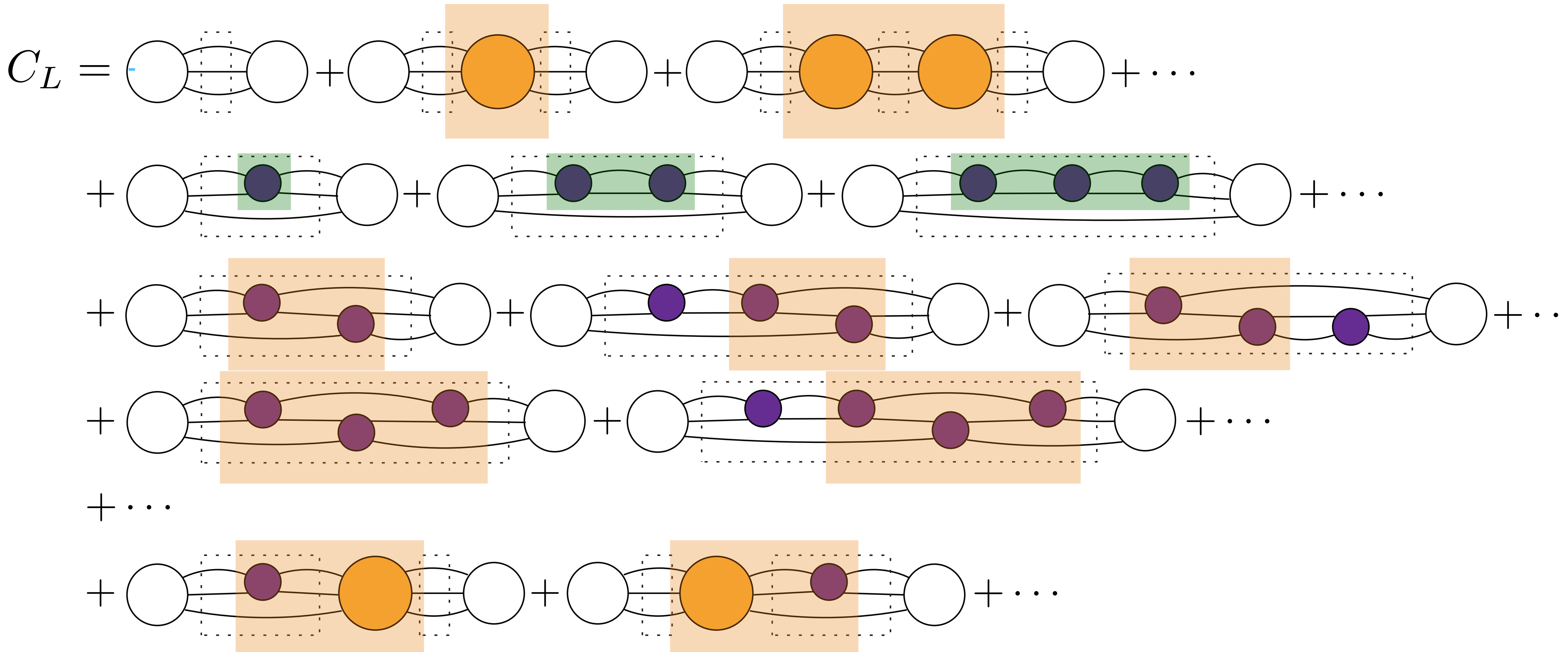
single **F** requires special treatment (1/3)

have not yet considered entire diagram contributions

missing contributions from ***off-shellness***

missing smooth terms (short-distance parts)

Short-distance parts & summation



$$C_L - C_\infty = \mathbf{A}'_3\,\mathbf{F}_{33}\,\mathbf{A}_3 + \mathbf{A}'_3\,\mathbf{F}_{33}\,\mathbf{K}_{\mathrm{df},3}\,\mathbf{F}_{33}\,\mathbf{A}_3 + \cdots$$

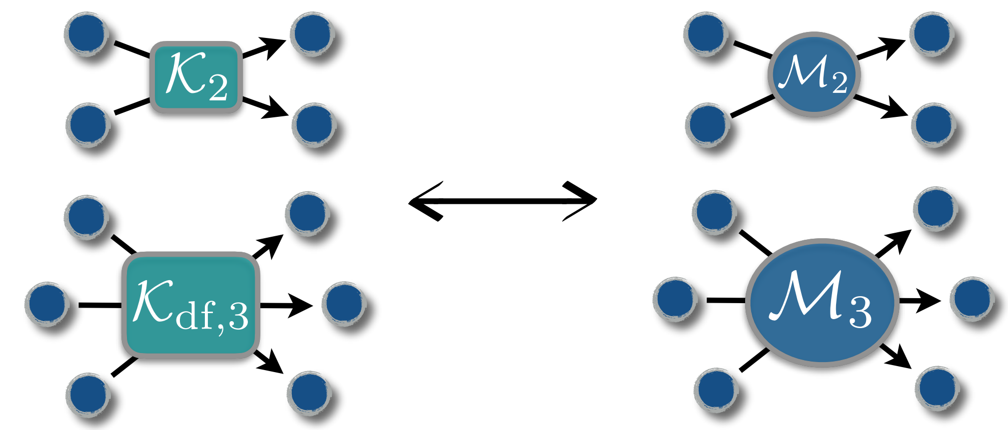
$$= \mathbf{A}'_3 \frac{1}{\mathbf{F}_{33}^{-1} + \mathbf{K}_{\mathrm{df},3}} \mathbf{A}_3$$

$$\mathbf{F}_{33} \equiv \frac{1}{3}\mathbf{F} + \mathbf{F}\,\mathbf{K}_2 \frac{1}{1 - (\mathbf{F}+\mathbf{G})\,\mathbf{K}_2}\mathbf{F}$$

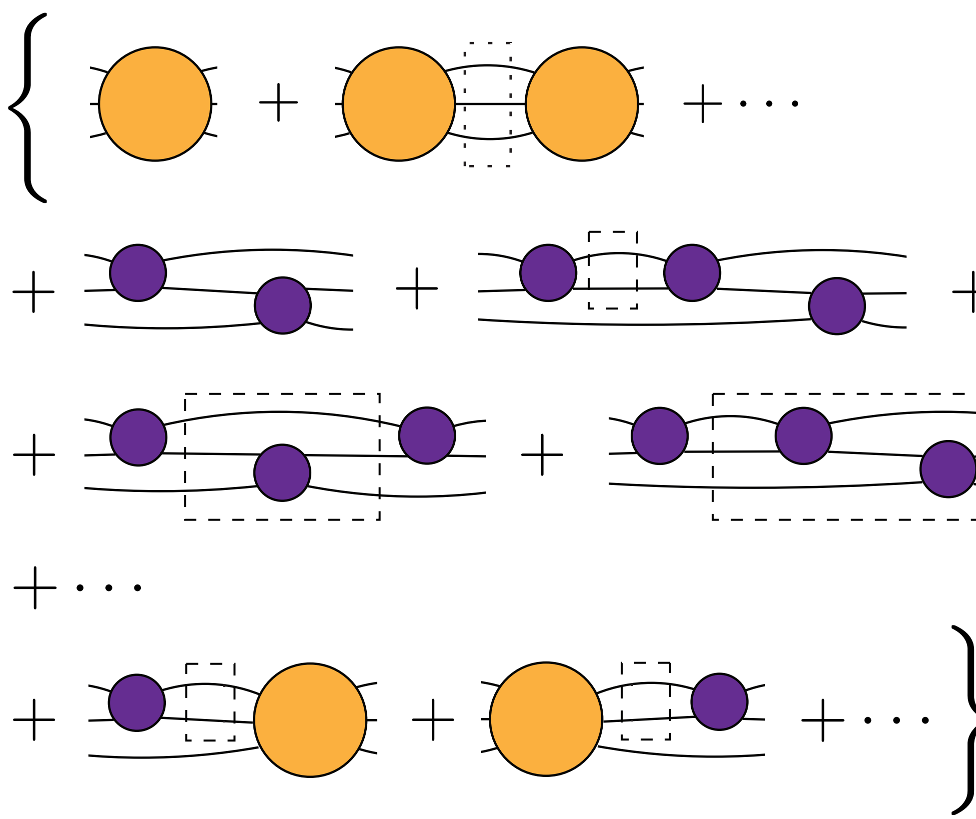
no term left behind

Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

- ☐ Define a new finite-volume correlator
  1. Amputate interpolating fields
  2. Drop disconnected diagrams
  3. Symmetrize



$$\mathcal{M}_{L,3} \equiv \mathcal{S}\{$$



$$\} = \mathcal{F}_L(\mathcal{K}_2, \mathcal{K}_{\mathrm{df},3})$$

known functional of $K_{df,3}$

$$\mathcal{M}_3 = \lim_{L\to\infty} \mathcal{F}_L(\mathcal{K}_2, \mathcal{K}_{\mathrm{df},3})$$

- MTH, Sharpe (2015)

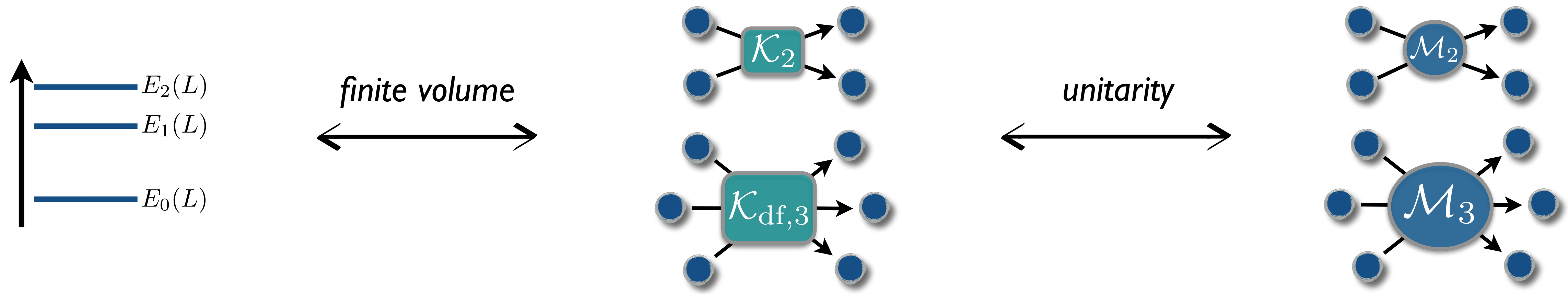
Integral equations, all inputs known

Exact unitarity solution

- Briceño, MTH, Sharpe, Szczepaniak (2019)

Result

$$\det[\mathcal{K}_{\mathrm{df},3}(s)^{-1} + F_3(\mathcal{M}_2, P, L)] = 0 \qquad\qquad \mathcal{M}_3 = \lim_{L\to\infty} \mathcal{F}_L(\mathcal{K}_2, \mathcal{K}_{\mathrm{df},3})$$

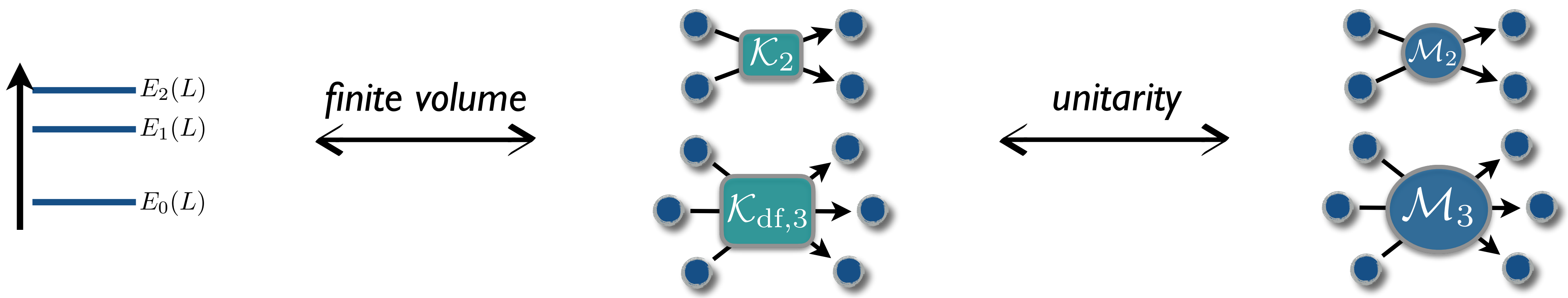


Holds only for three-particle energies $s < (5m)^2$

Neglects e^{-mL}

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
Li, Liu (2013) • Briceño (2014)

Status…



- [x] Identical spin-zero, no 2-to-3, no K2 poles • MTH, Sharpe (2014, 2015) •
- [x] as above… but including 2-to-3 • Briceño, MTH, Sharpe (2017) •
- [x] as above… but including K2 poles • Briceño, MTH, Sharpe (2018) •
- [x] Non-identical, non-degenerate spin-zero $\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$
  • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020, 2021)
- [ ] Multiple three-particle channels… Spin!

Related work

- Finite-volume unitarity method

  Döring, Mai (2016,2017)

  Gives connection to unitarity relations

- Non-relativistic EFT method

  Hammer, Pang, Rusetsky (2017)

  Simplified derivation + integral equations

  Do not yet include non-degenerate, 2-to-3

- All methods

  Rely on intermediate, scheme-dependent quantity

  Hold up to e^{-mL} and for $E_3^\star < 5m_\pi$

  Equivalent where comparable

- Review articles

  MTH and Sharpe, 1901.00483 • Rusetsky, 1911.01253 • Mai, Döring, Rusetsky, 2103.00577

Not covered here

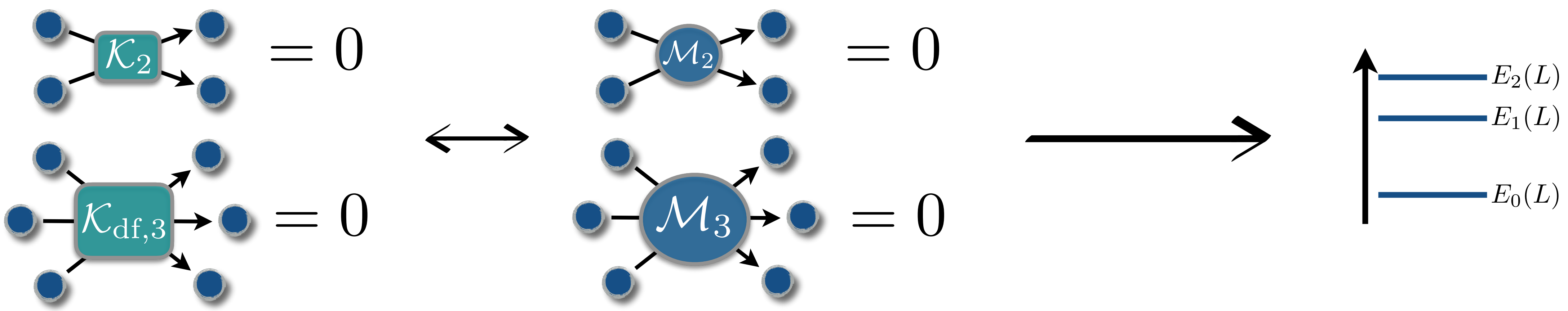
- Activity extracting and fitting three-hadron energies

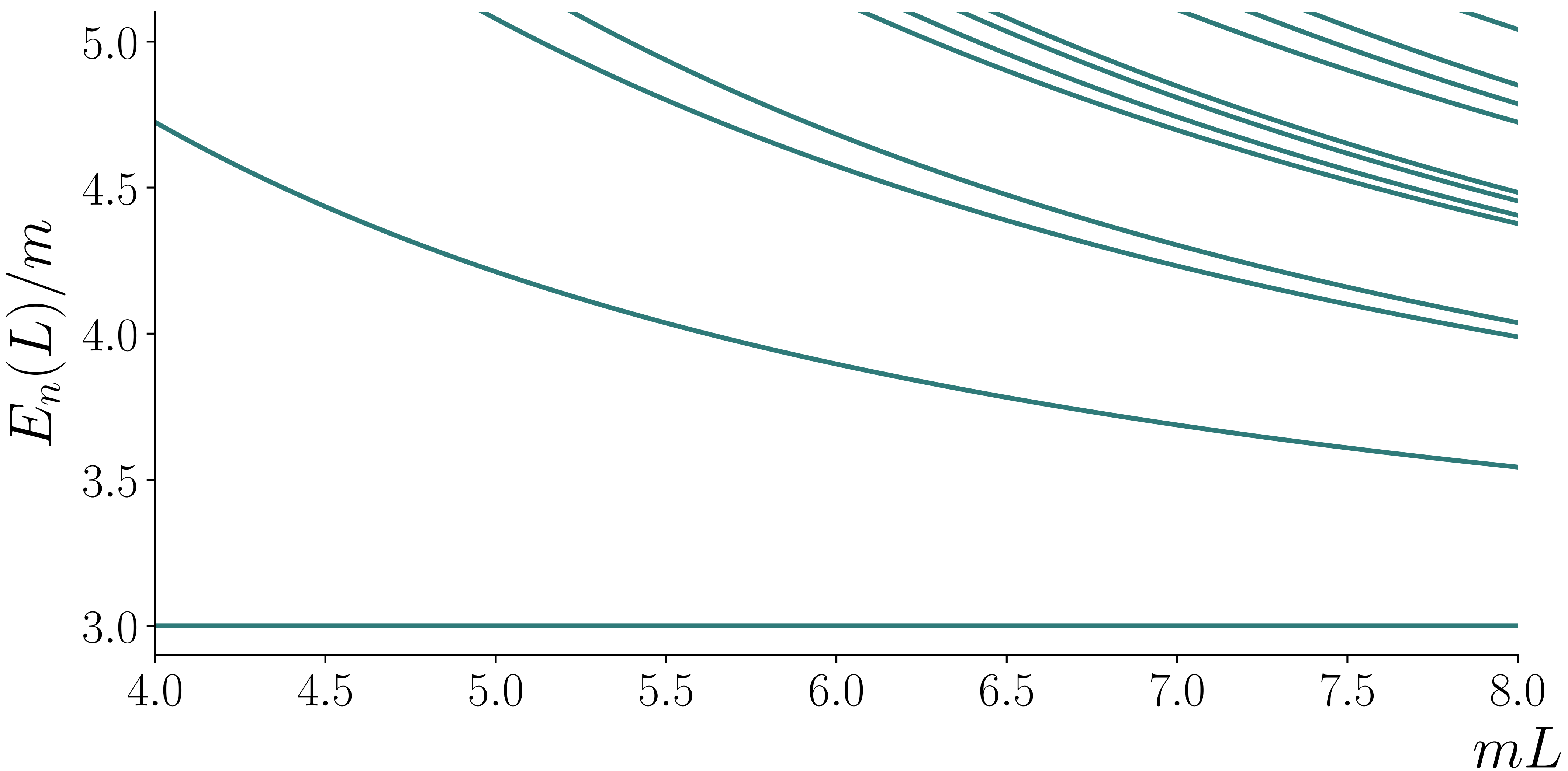
• Hörz, Hanlon (2019) • Blanton, Romero-López, Sharpe (2019) •
• Alexandru, Brett, Culver, Döring, Guo, Lee, Mai (2019) •
• Fischer, Kostrzewa, Liu, Romero-López, Ueding, Urbach (2020) •
• Mai, Alexandru, Brett, Culver, Döring, Lee, Sadasivan (2021) •
• Blanton, Hanlon, Hörz, Morningstar, Romero-López, Sharpe (2021) •

- Activity connecting and extending formalisms

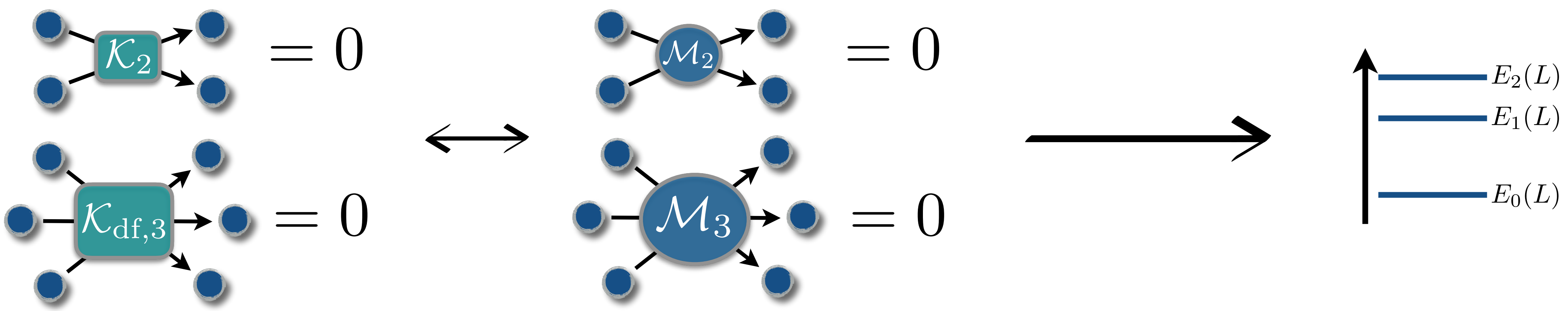
Relating infinite-volume equations	• Jackura *et al.* (2019) •
Alternative derivations	• Blanton, Sharpe (2020) •
Equivalence of formalisms (where comparable)	• Blanton, Sharpe (2020) •

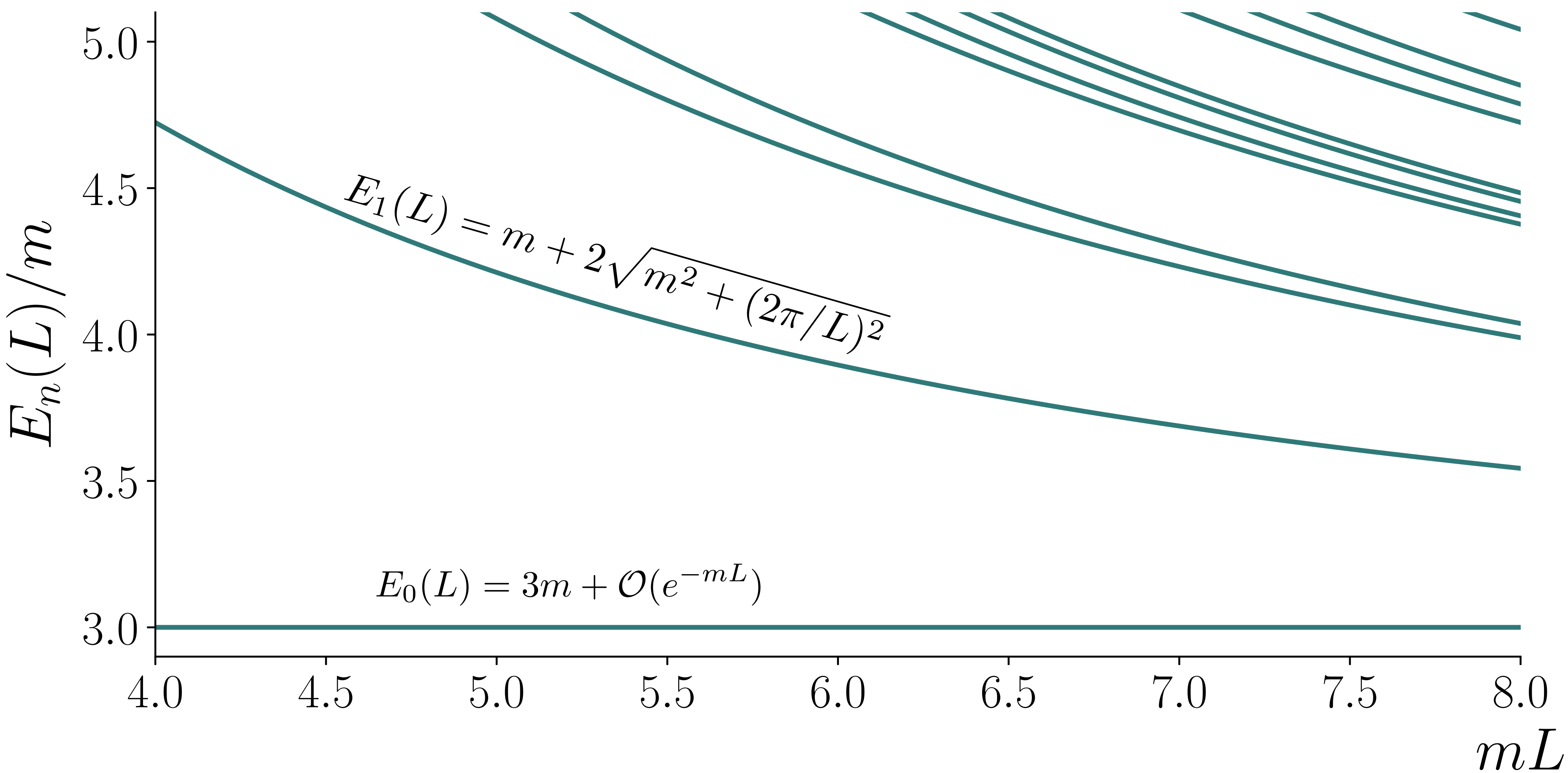
Non-interacting energies



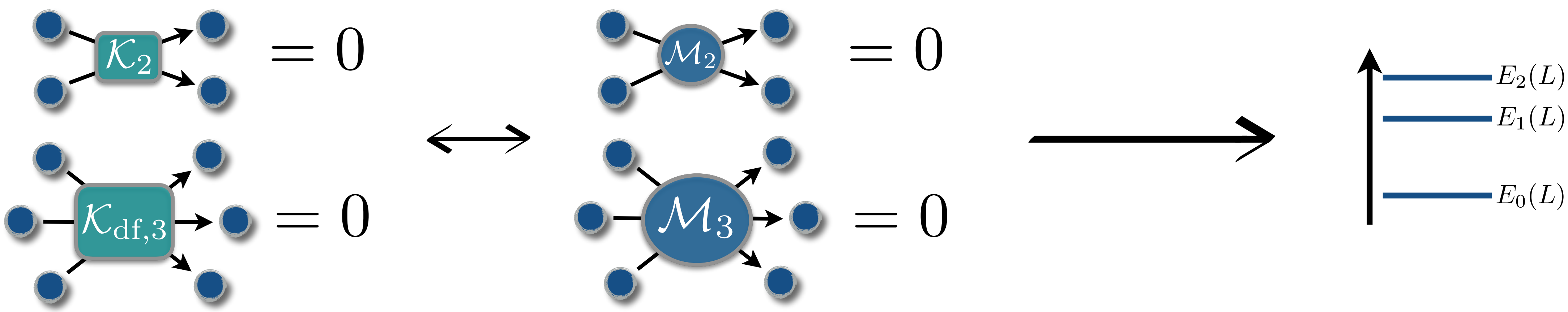


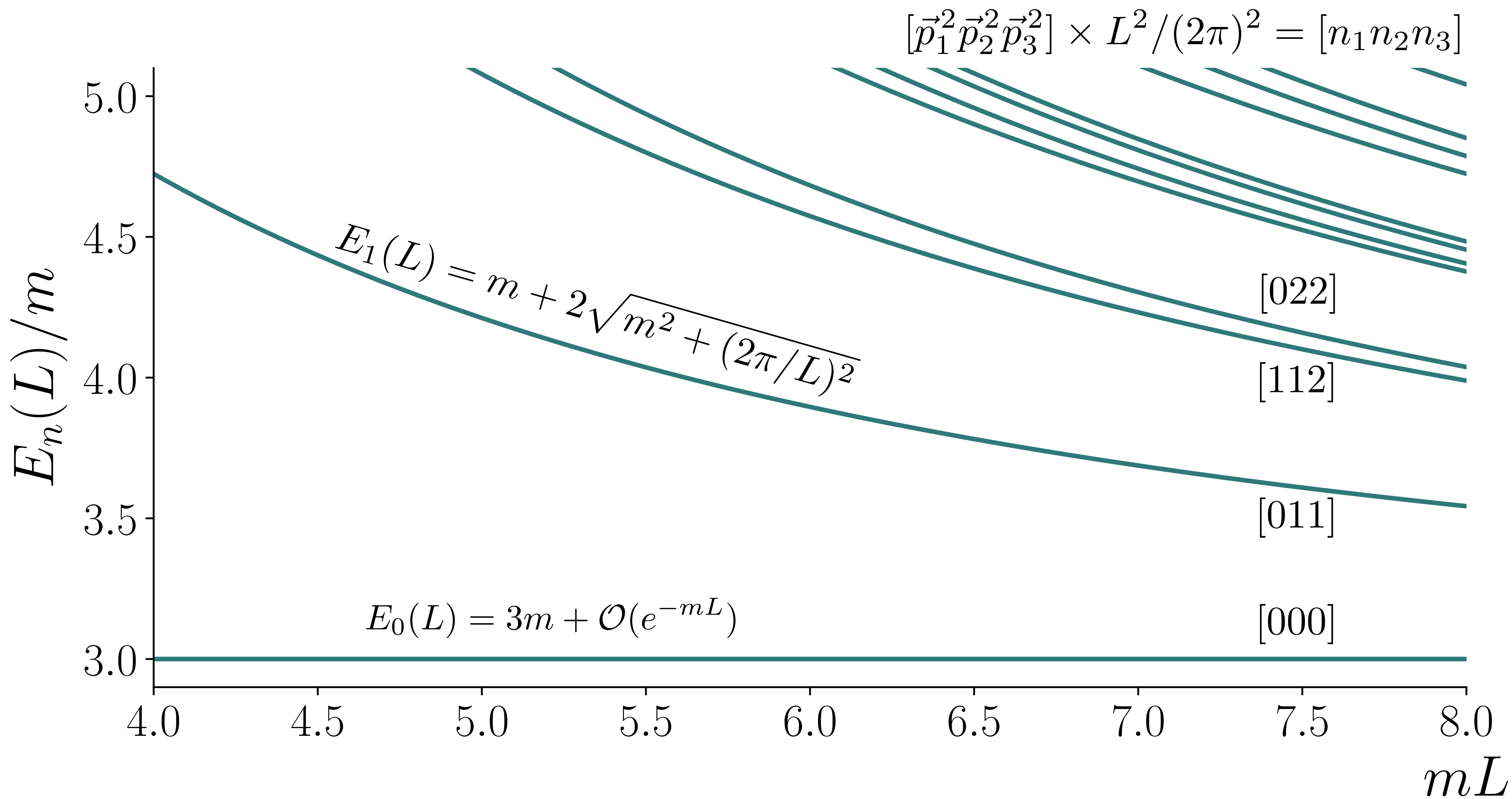
Non-interacting energies



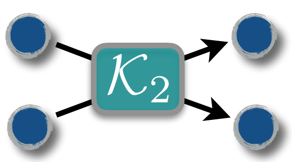


Non-interacting energies

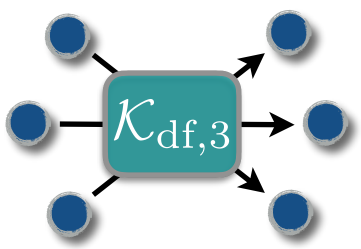




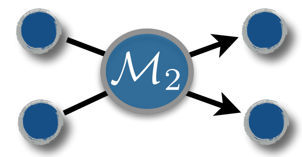
Two-particle interactions



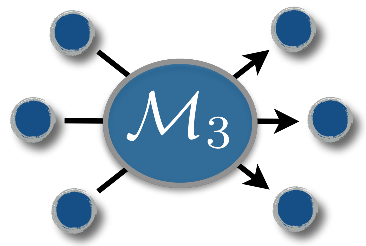
$= -16\pi\sqrt{s}\, a$



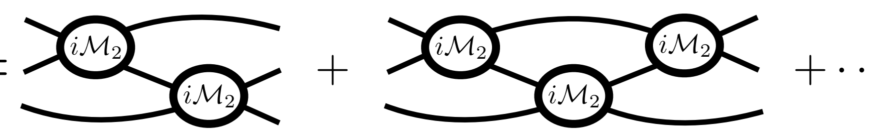
$= 0$

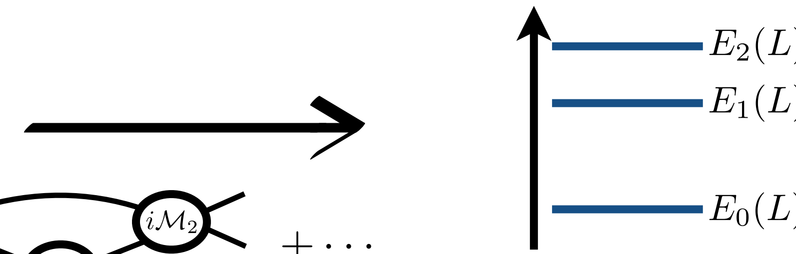


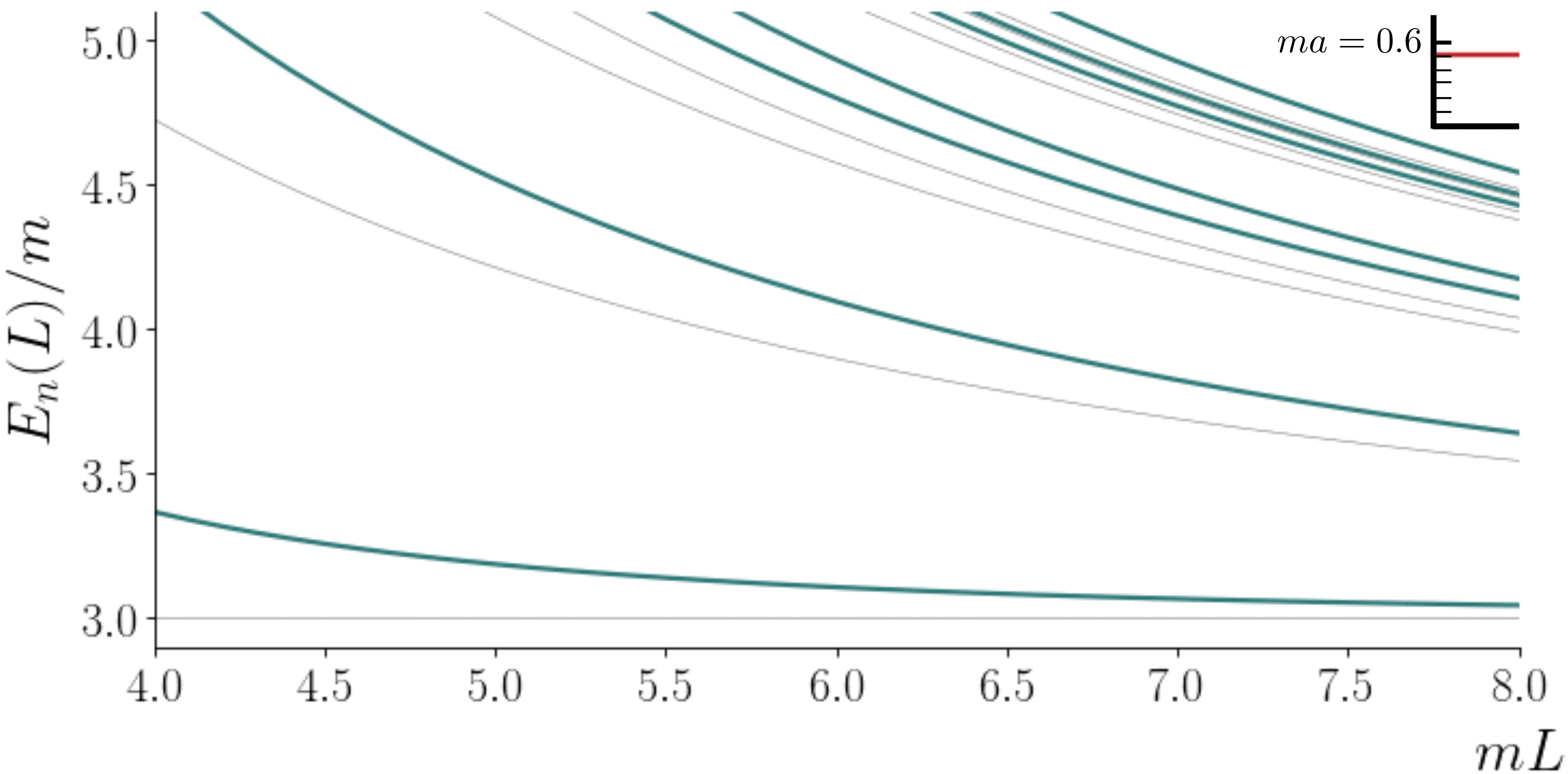
$$= \frac{16\pi\sqrt{s}}{-1/a - ip}$$



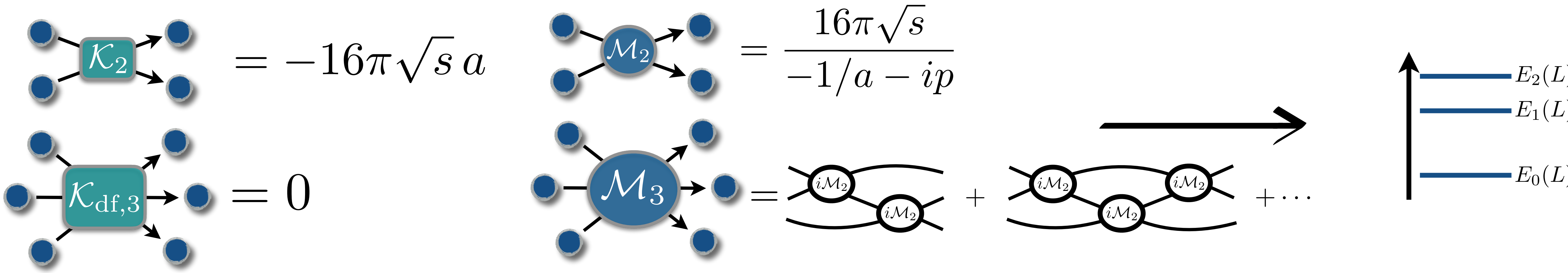
$=$

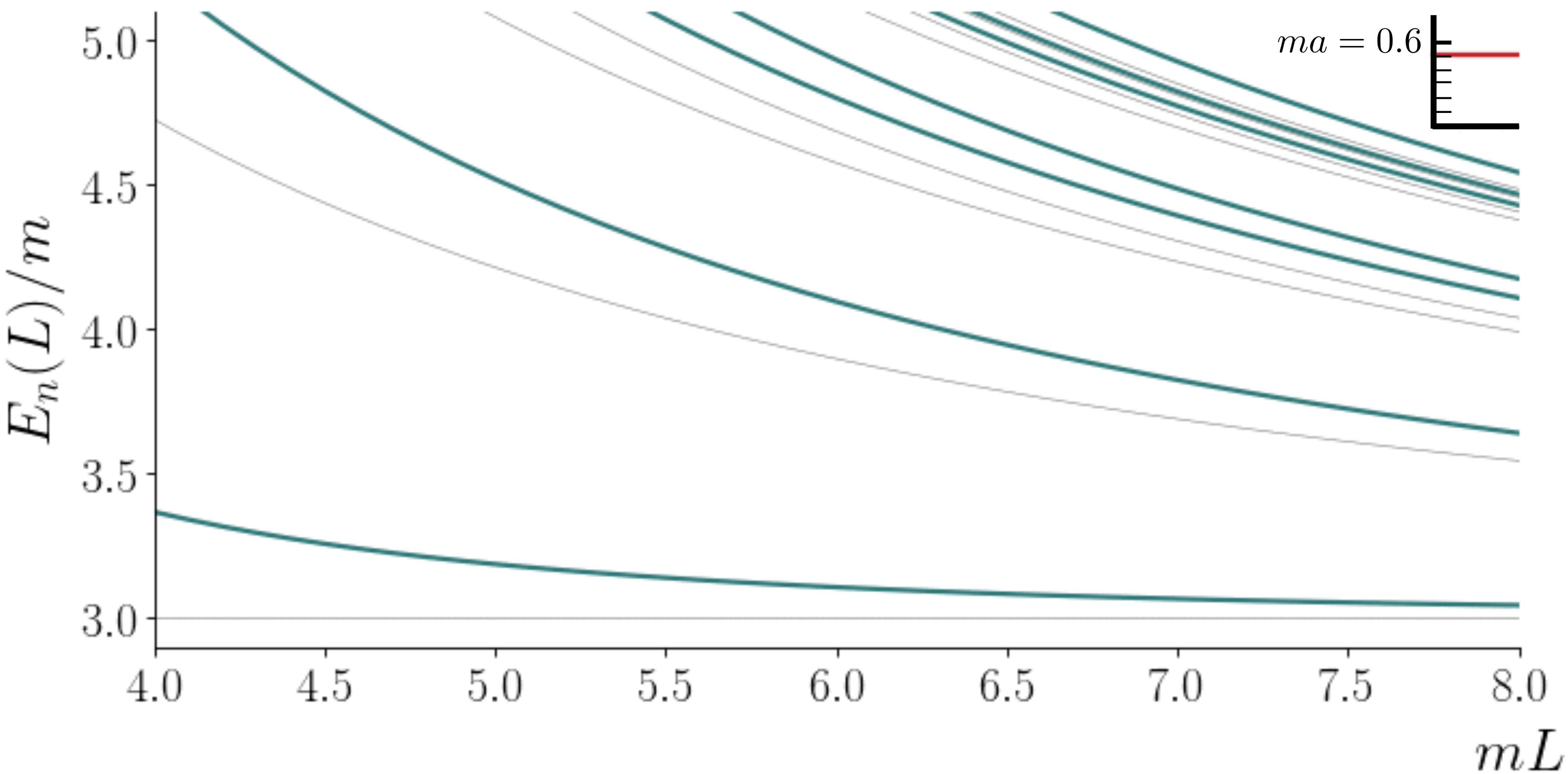






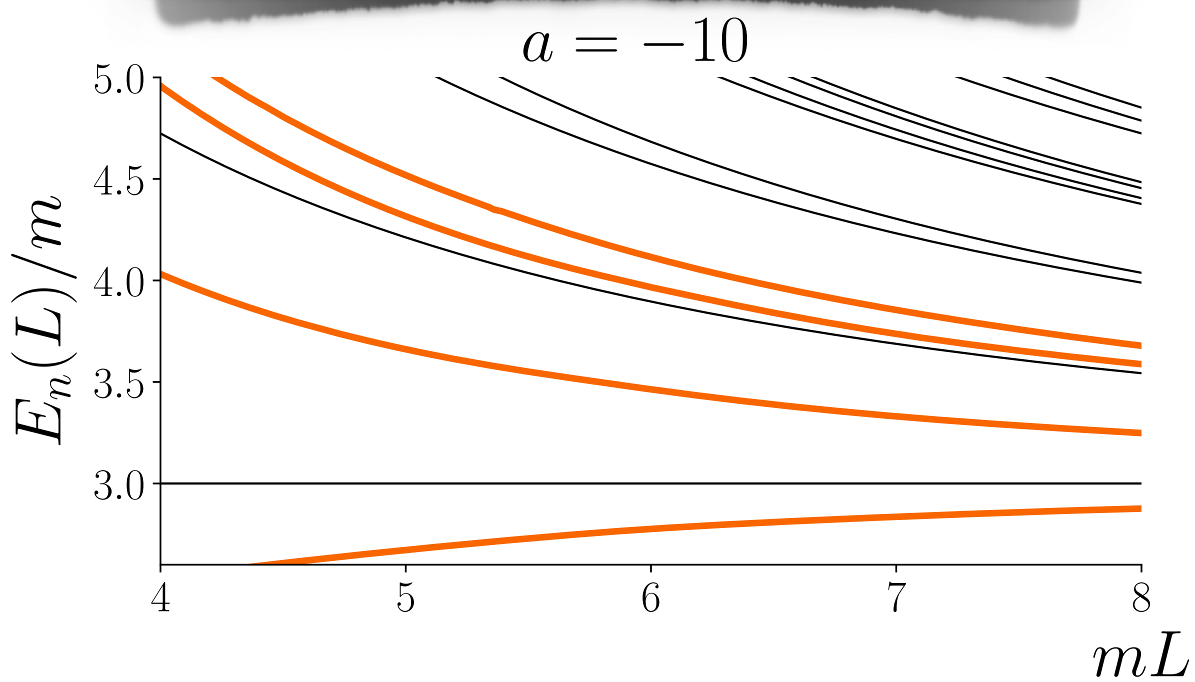
Two-particle interactions



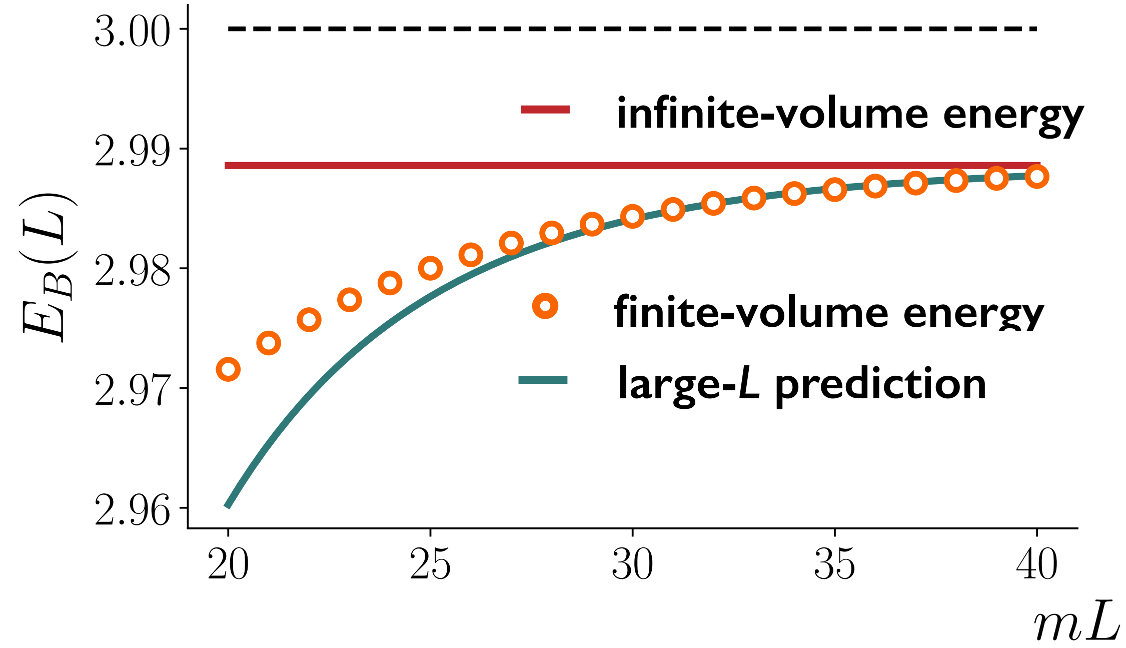


Checks and explorations

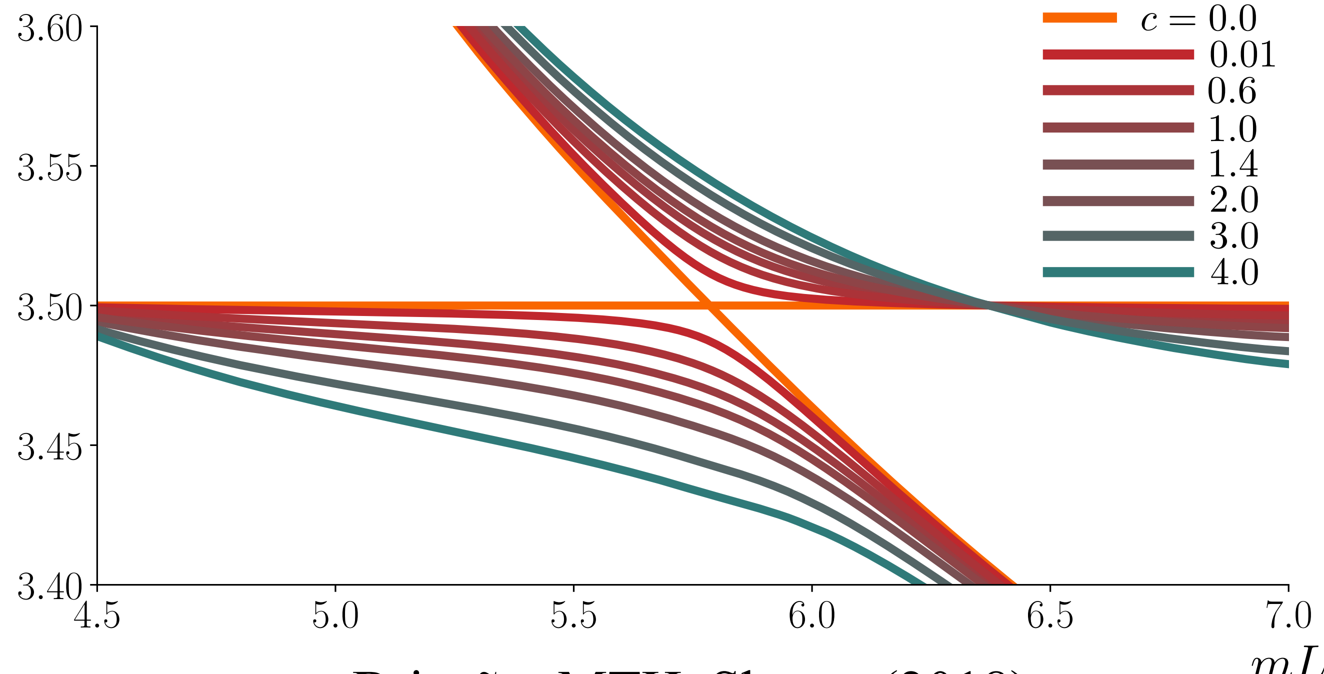
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



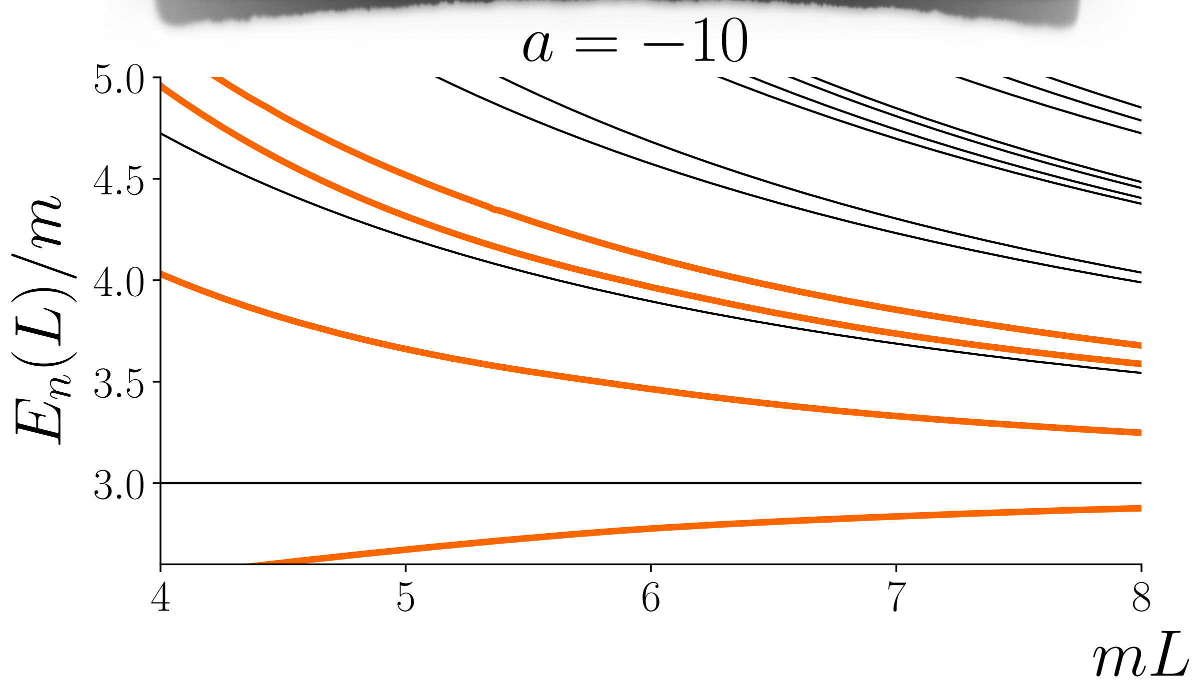
Model of a 3-particle resonance



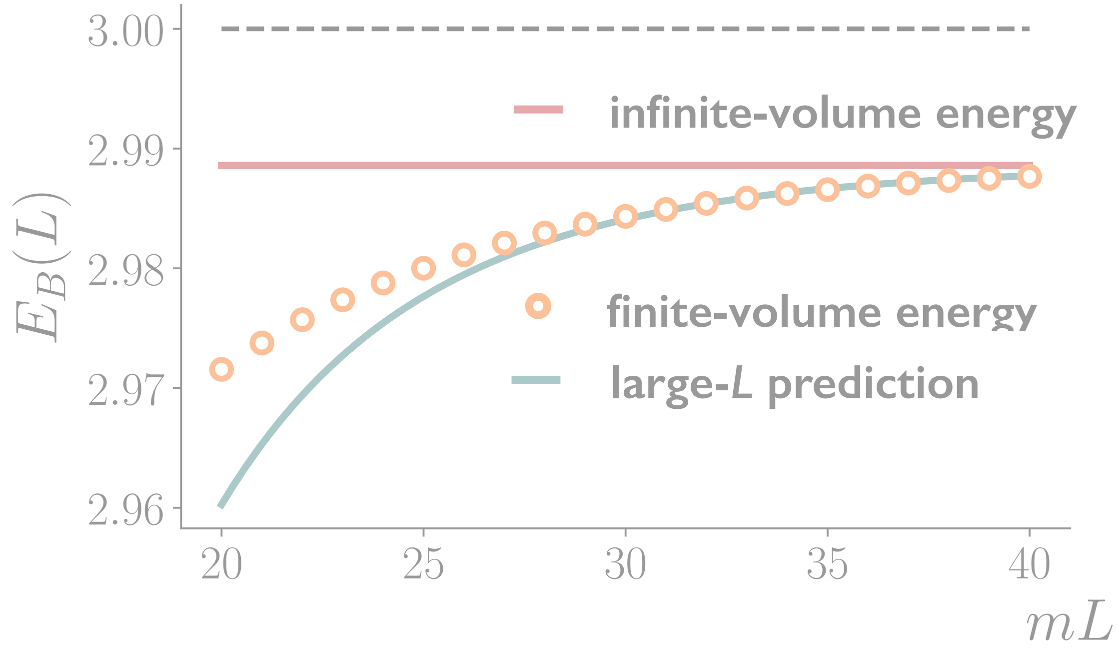
• Briceño, MTH, Sharpe (2018) •

Checks and explorations

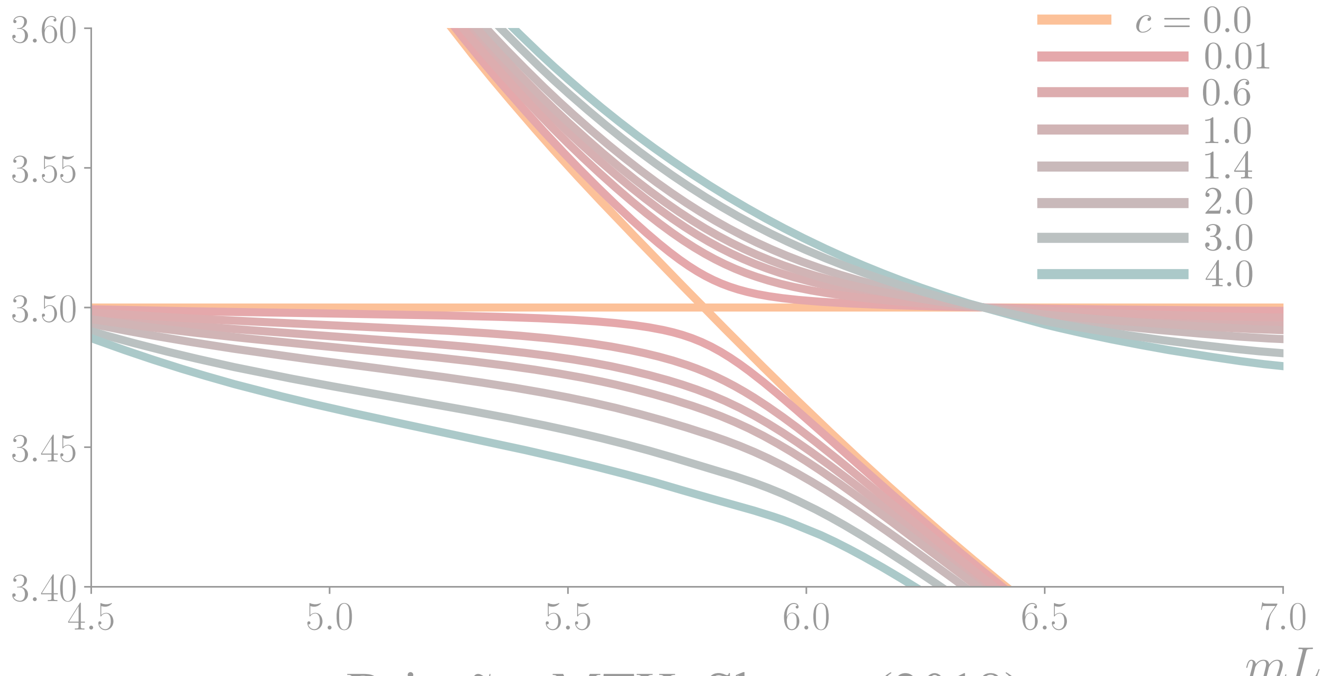
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



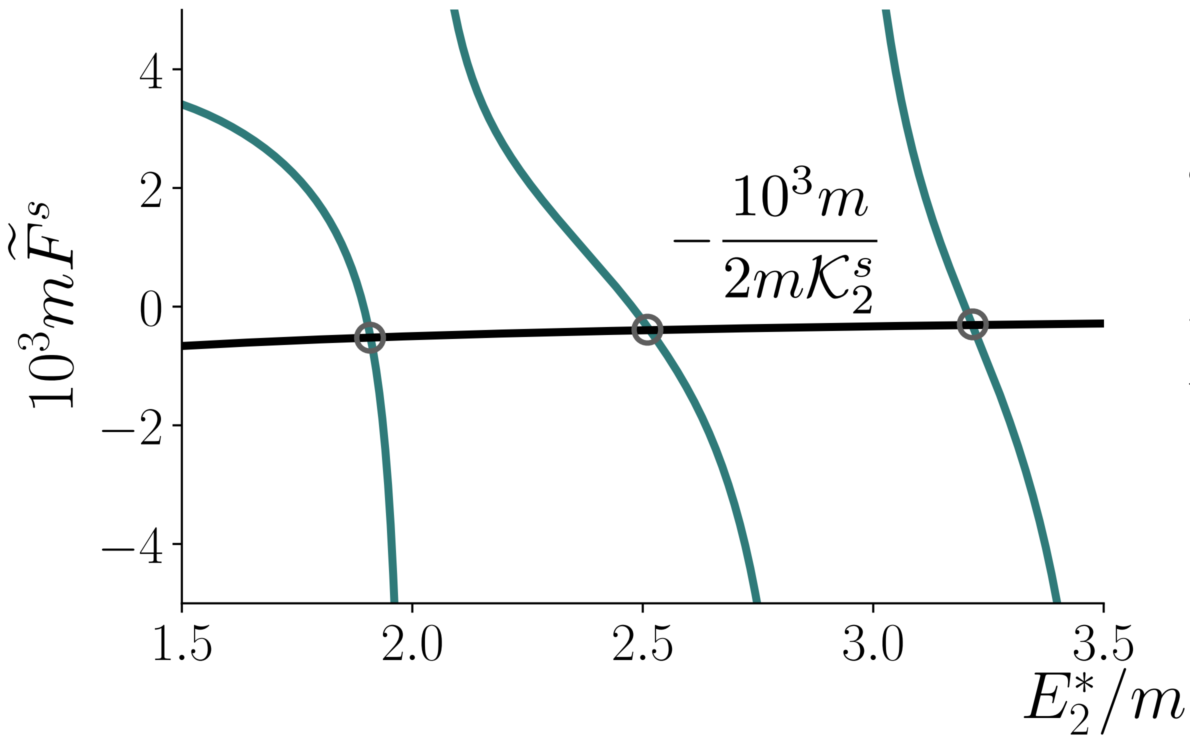
Model of a 3-particle resonance

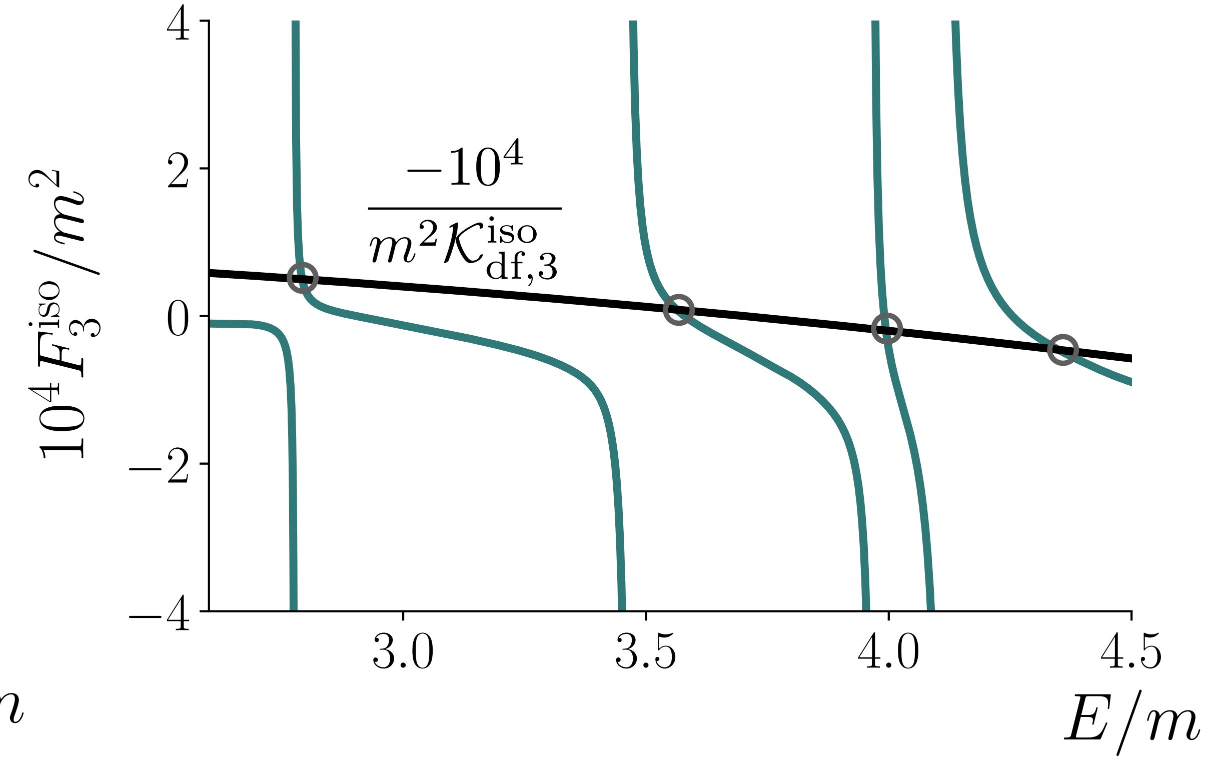


• Briceño, MTH, Sharpe (2018) •

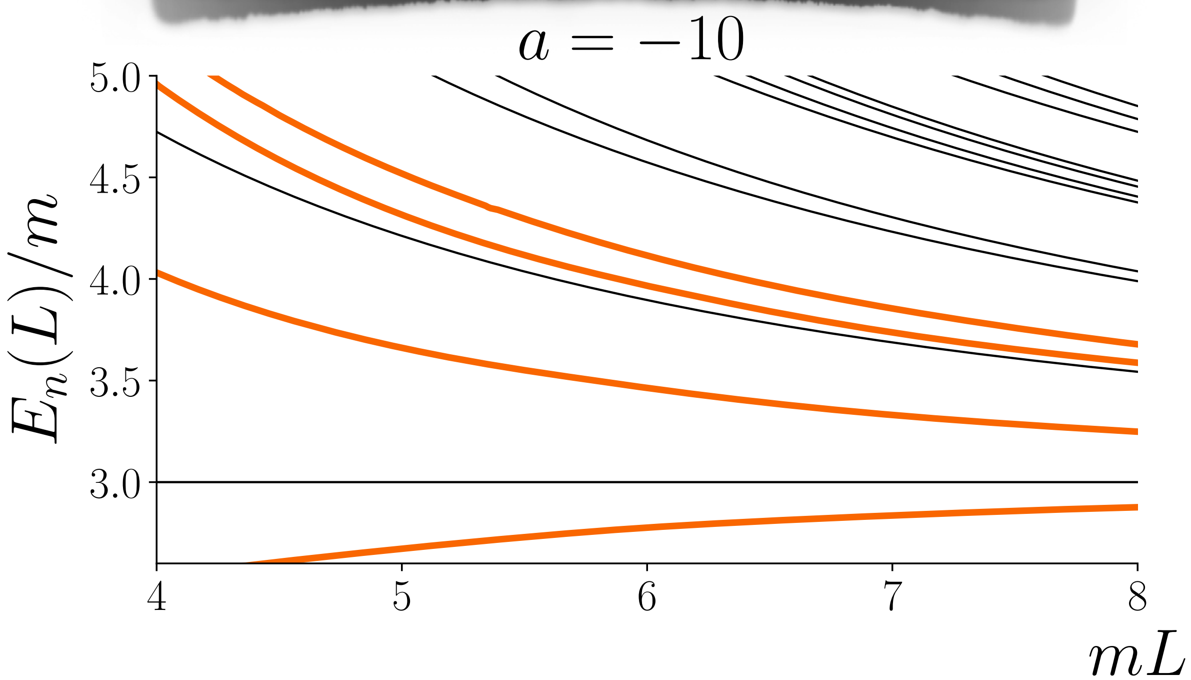
Attractive scattering

$$ma = -10\,,\ mL = 6$$



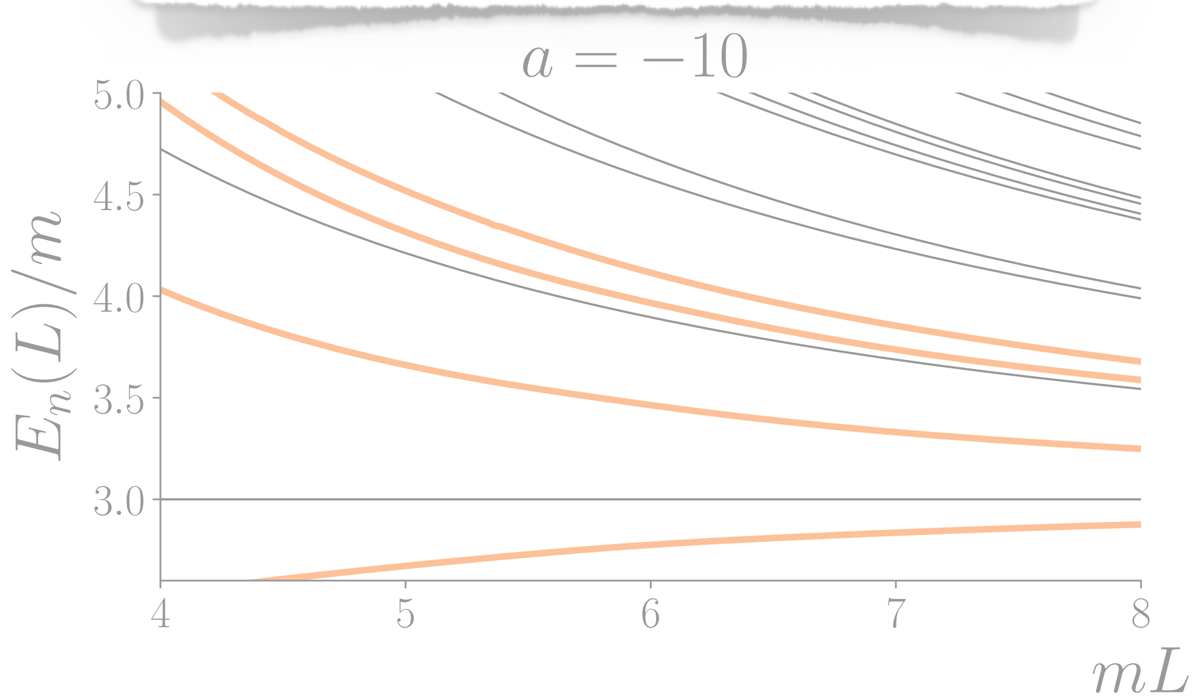


Spectrum with no 3-particle interaction

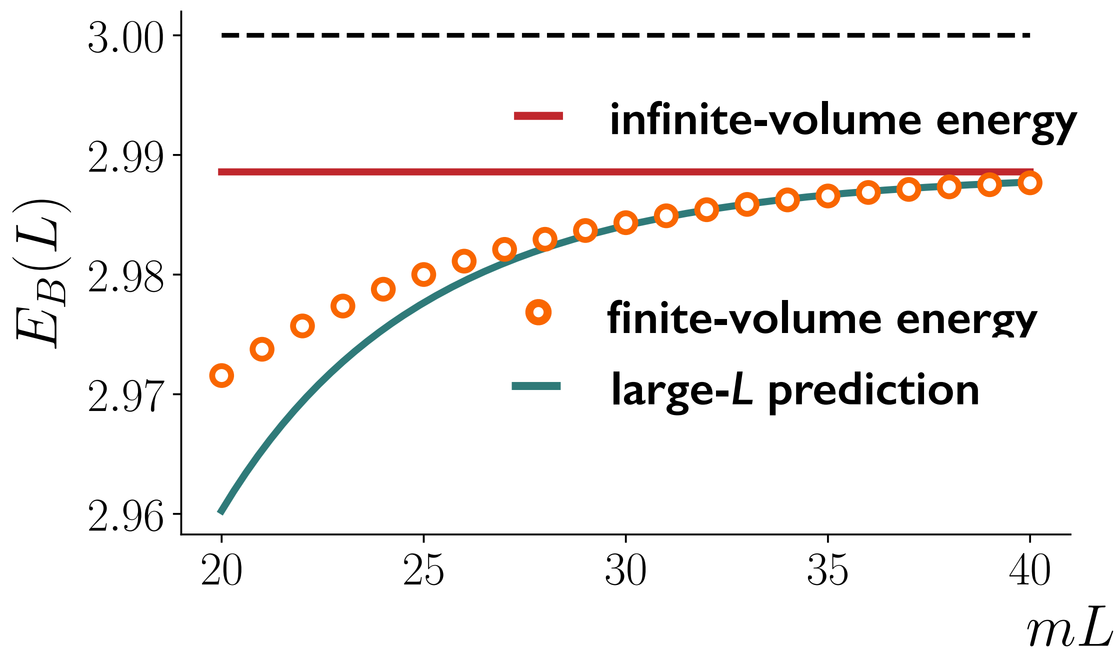


Checks and explorations

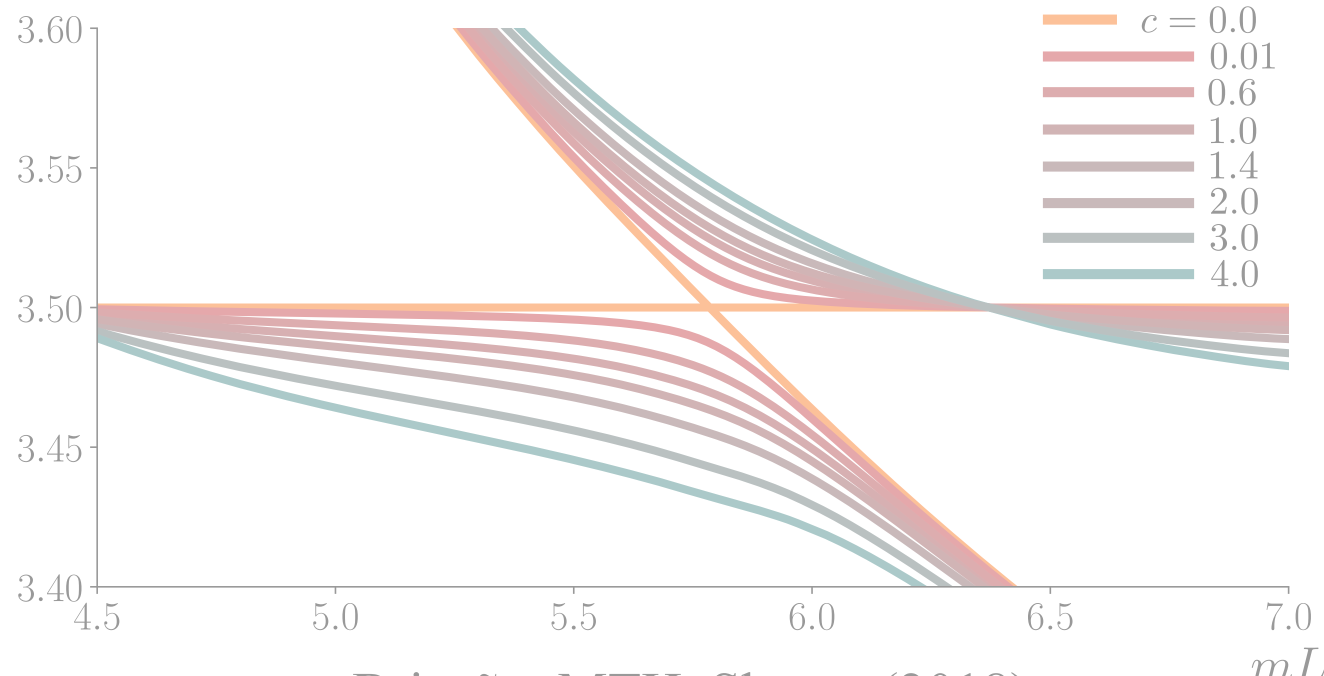
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



Model of a 3-particle resonance



- Briceño, MTH, Sharpe (2018)

Unitary bound state

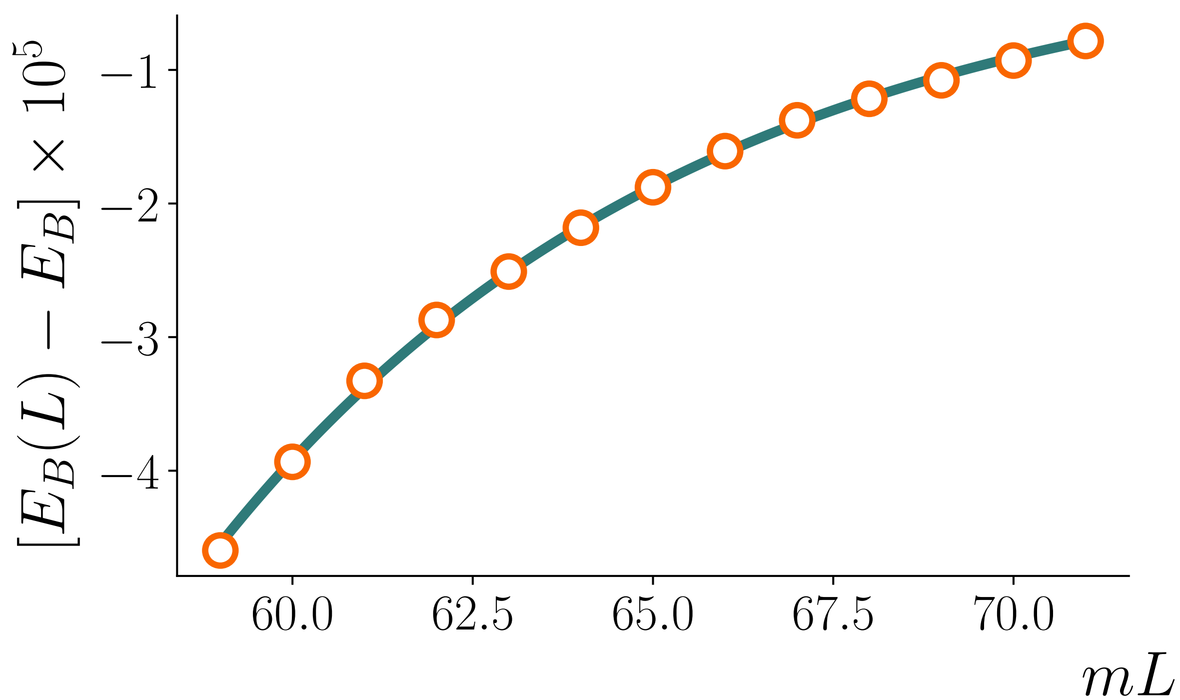
The parameters $a = -10^4$, $\mathcal{K}^{\rm iso}_{\rm df,3}(E) = 2500$ lead to a shallow bound state

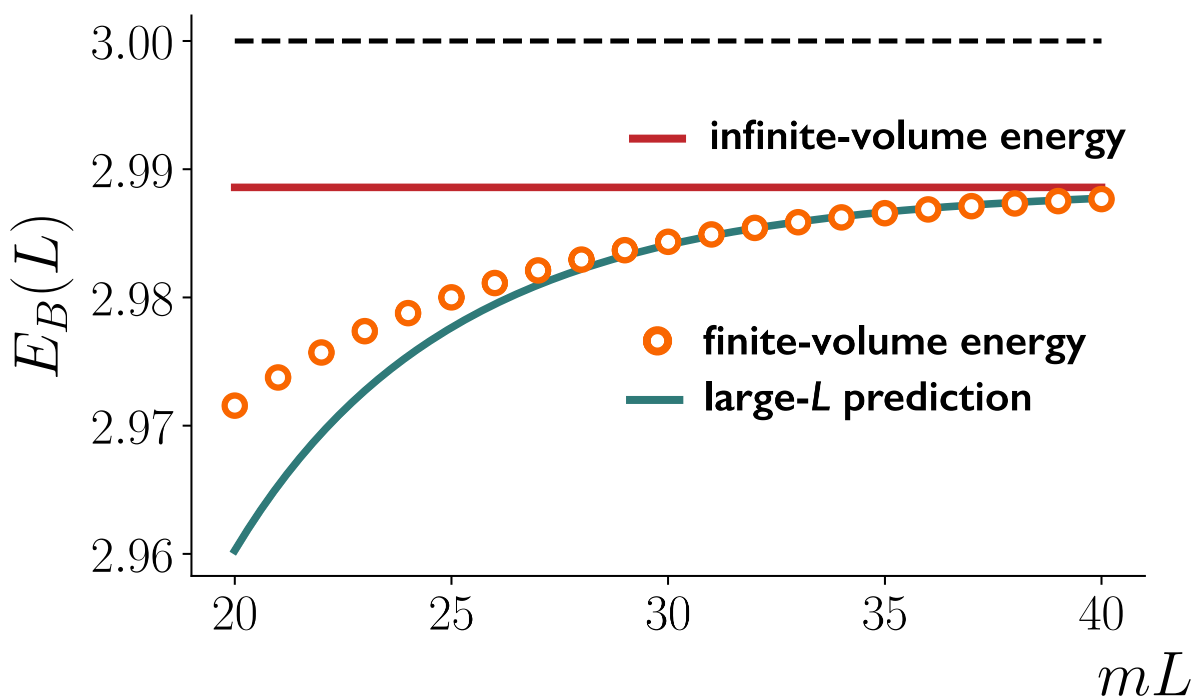
$\kappa \approx 0.1m$ where $E_B = 3m - \kappa^2/m$

Finite-volume behavior of this state has a known asymptotic form

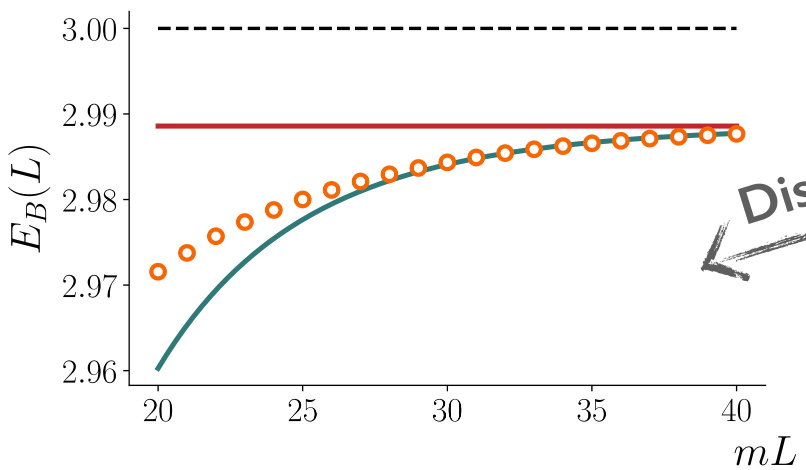
$$E_B(L) = 3m - \frac{\kappa^2}{m} - (98.35\cdots)|A|^2\frac{\kappa^2}{m}\frac{e^{-2\kappa L/\sqrt{3}}}{(\kappa L)^{3/2}}\left[1 + \mathcal{O}\left(\frac{1}{\kappa L}, \frac{\kappa^2}{m^2}, e^{-\alpha\kappa L}\right)\right]$$

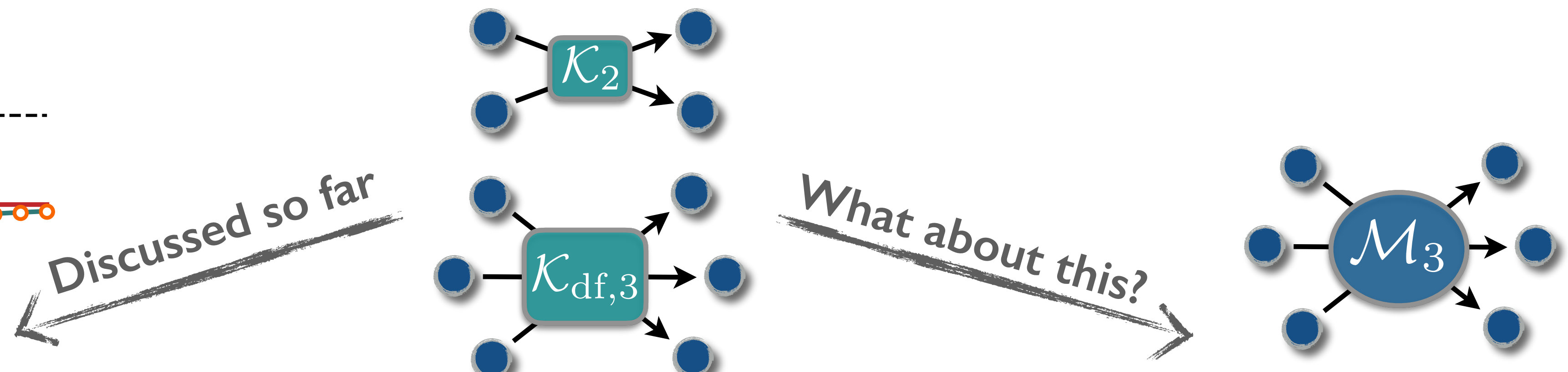
• Meißner, Rios, Rusetsky (2015) •



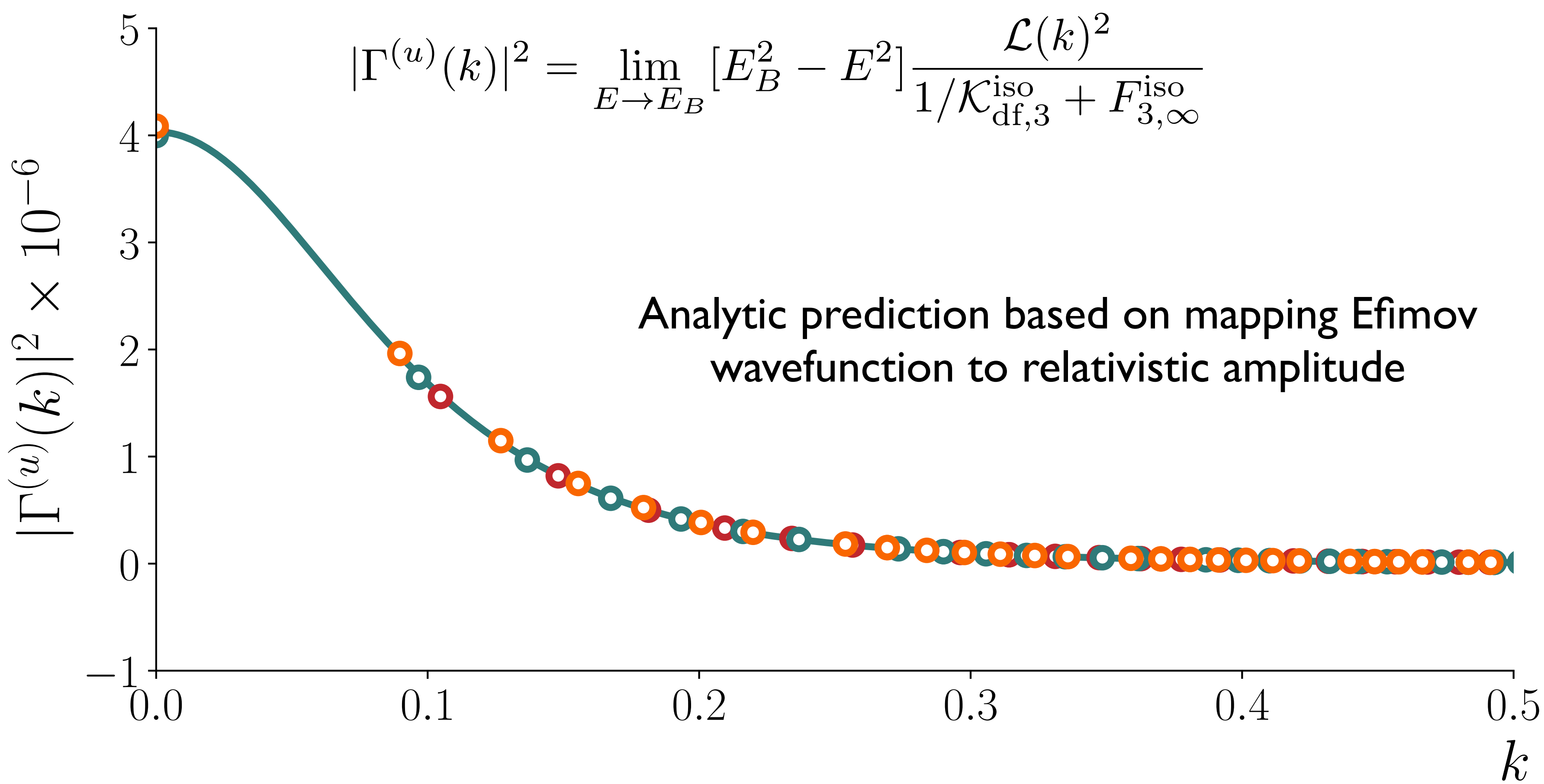


Bound state scattering amplitude

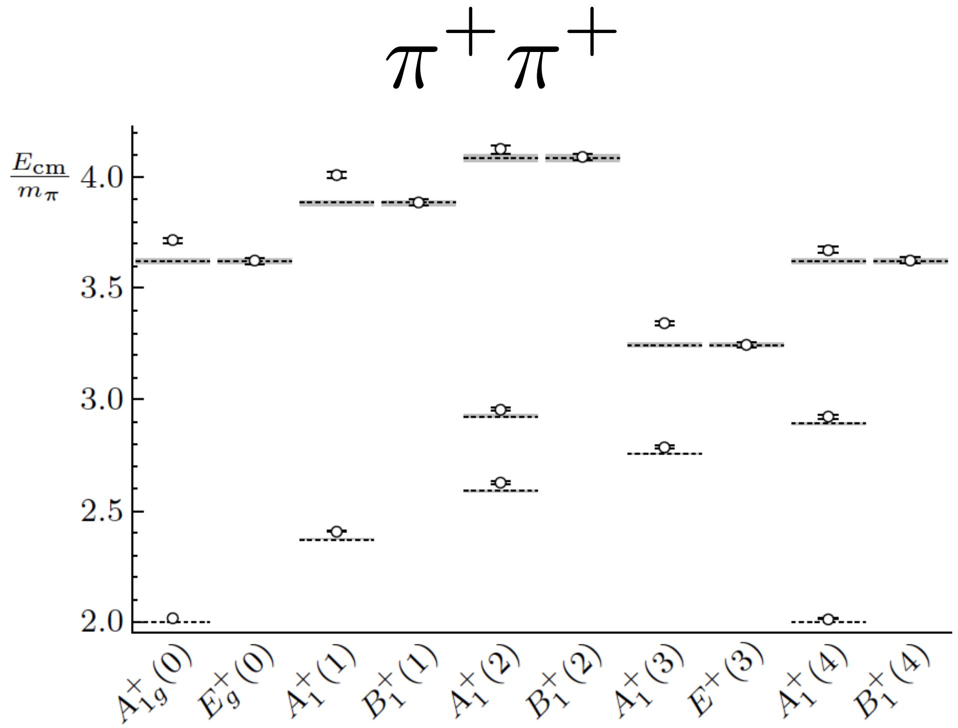


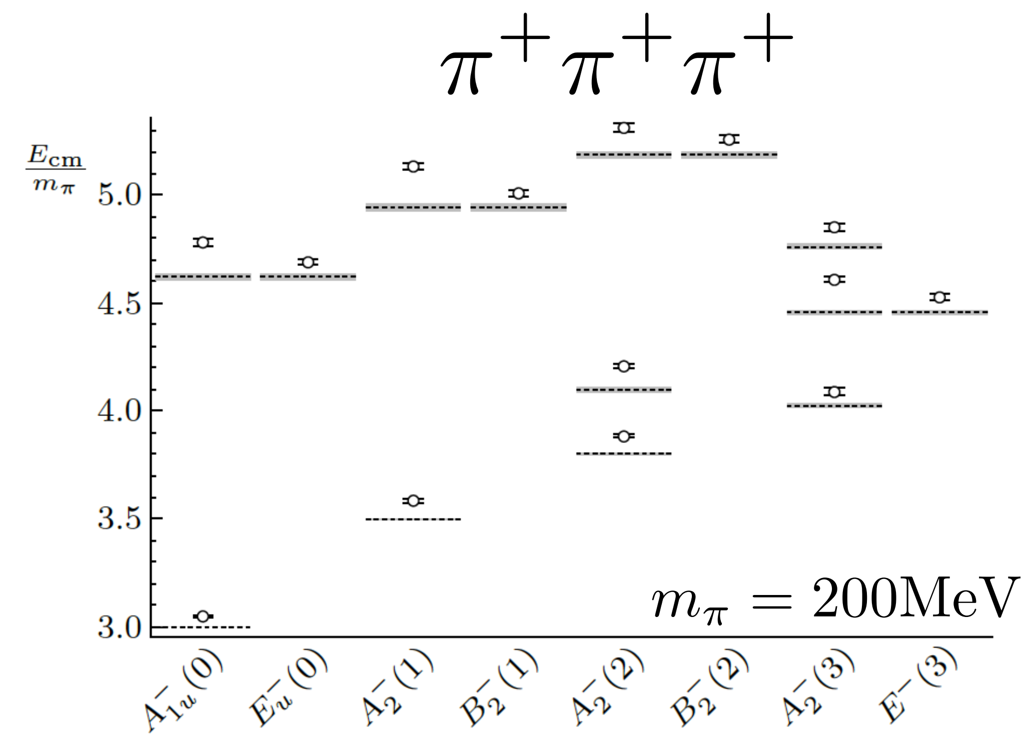


$$\mathcal{M}_3 \sim -\frac{\Gamma(\Omega_3')\overline{\Gamma}(\Omega_3)}{E^2 - M_B^2}$$

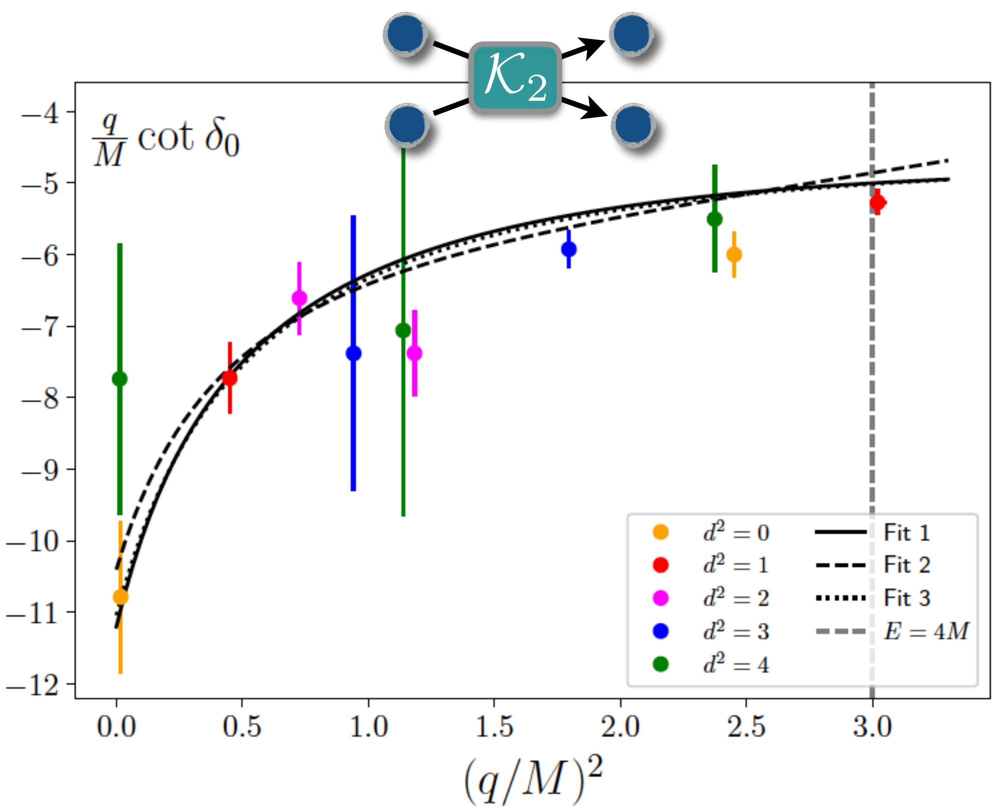


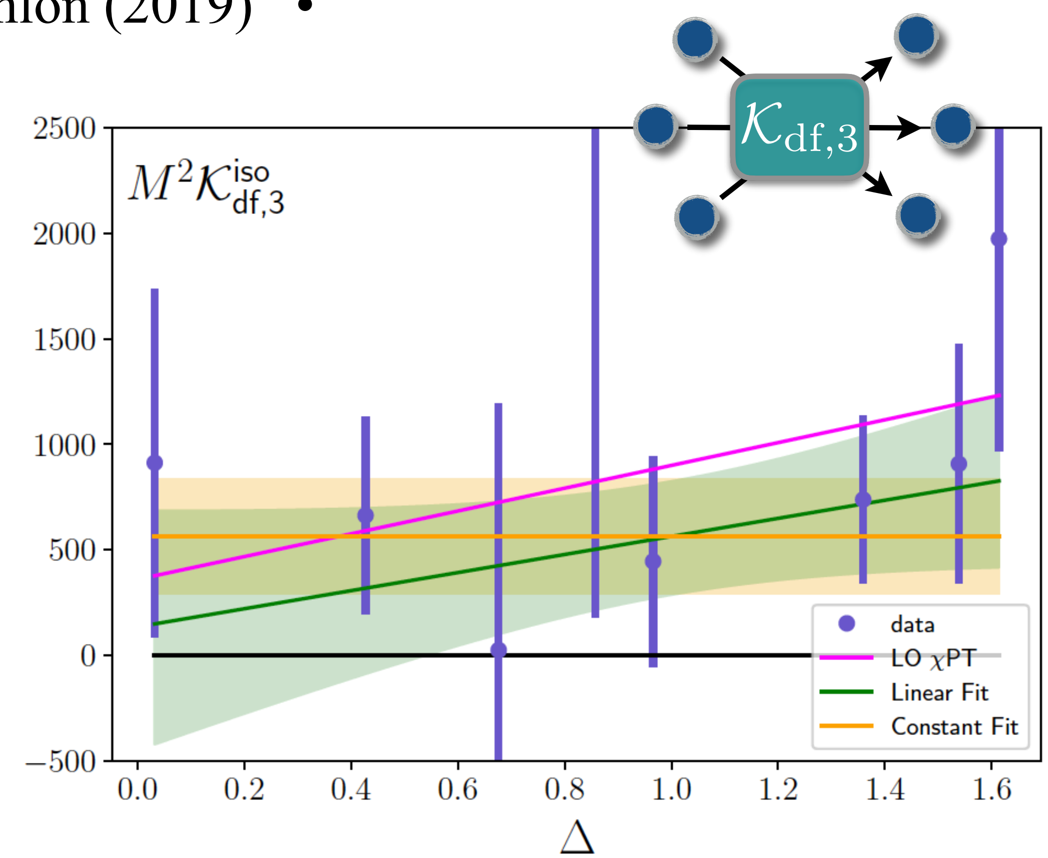
Lattice QCD *(CLS)*





• Hörz and Hanlon (2019) •



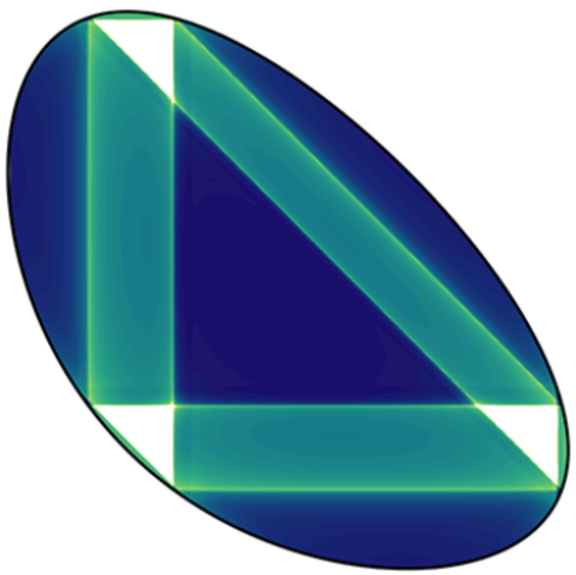


• Blanton, Romero-López, Sharpe (2019) • *see also Mai, Döring (2019)* •

Energy-Dependent $\pi^+\pi^+\pi^+$ Scattering Amplitude from QCD

Maxwell T. Hansen,[1,2,*] Raul A. Briceño,[3,4,†] Robert G. Edwards,[3,‡]
Christopher E. Thomas,[5,§] and David J. Wilson[5,∥]

(for the Hadron Spectrum Collaboration)



EDITORS' SUGGESTION

Energy-Dependent $\pi^+ \pi^+ \pi^+$ Scattering Amplitude from QCD

A three-hadron scattering amplitude is computed using lattice QCD for the first time.

Maxwell T. Hansen *et al.*
Phys. Rev. Lett. **126**, 012001 (2021)

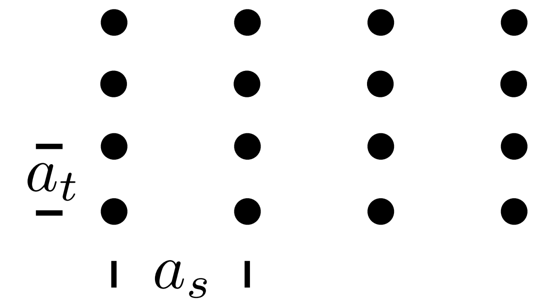



$$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$$

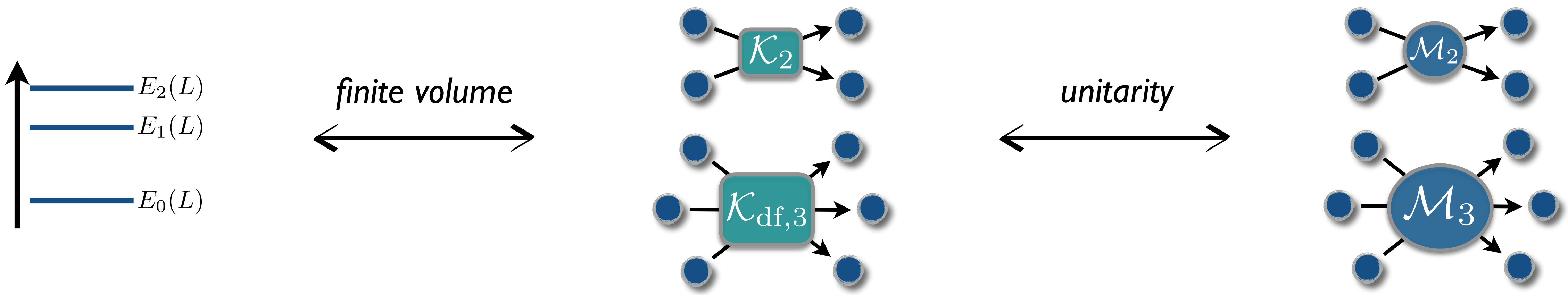
lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$ $\quad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$



❑ Workflow outline

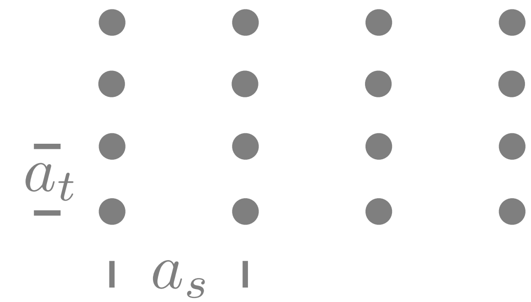


$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

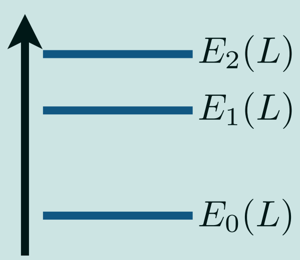
lattice details

$N_f = 2+1 \qquad a_s/a_t = 3.444(6) \qquad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV} \qquad a_s \approx 0.12\text{fm}$

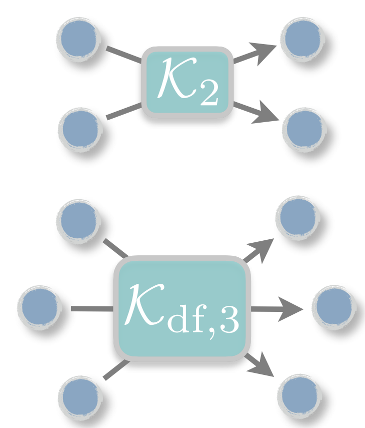


☐ Workflow outline

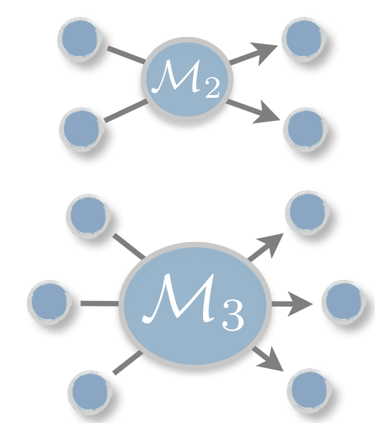


Determine finite-volume energy spectrum

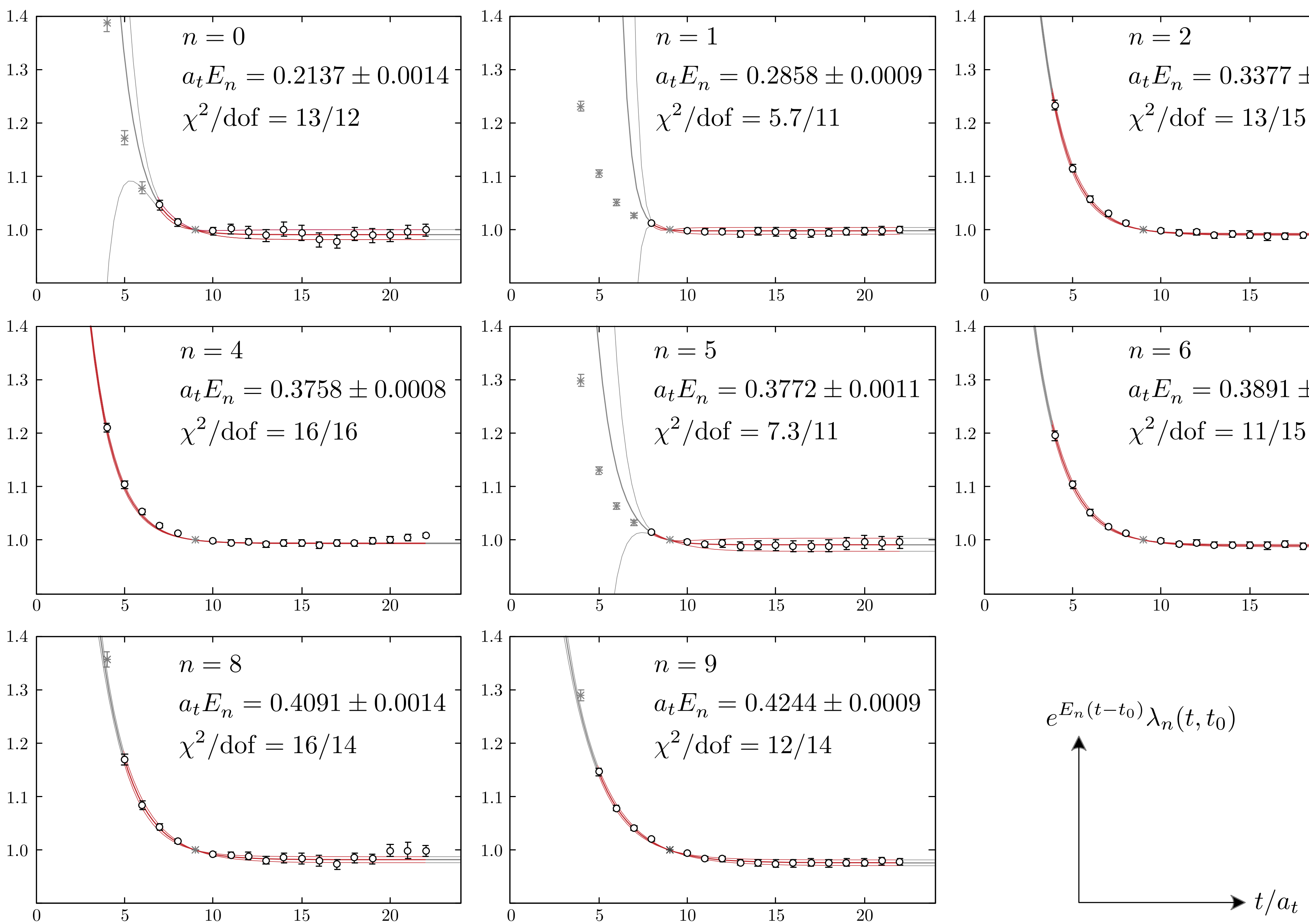
finite volume



unitarity



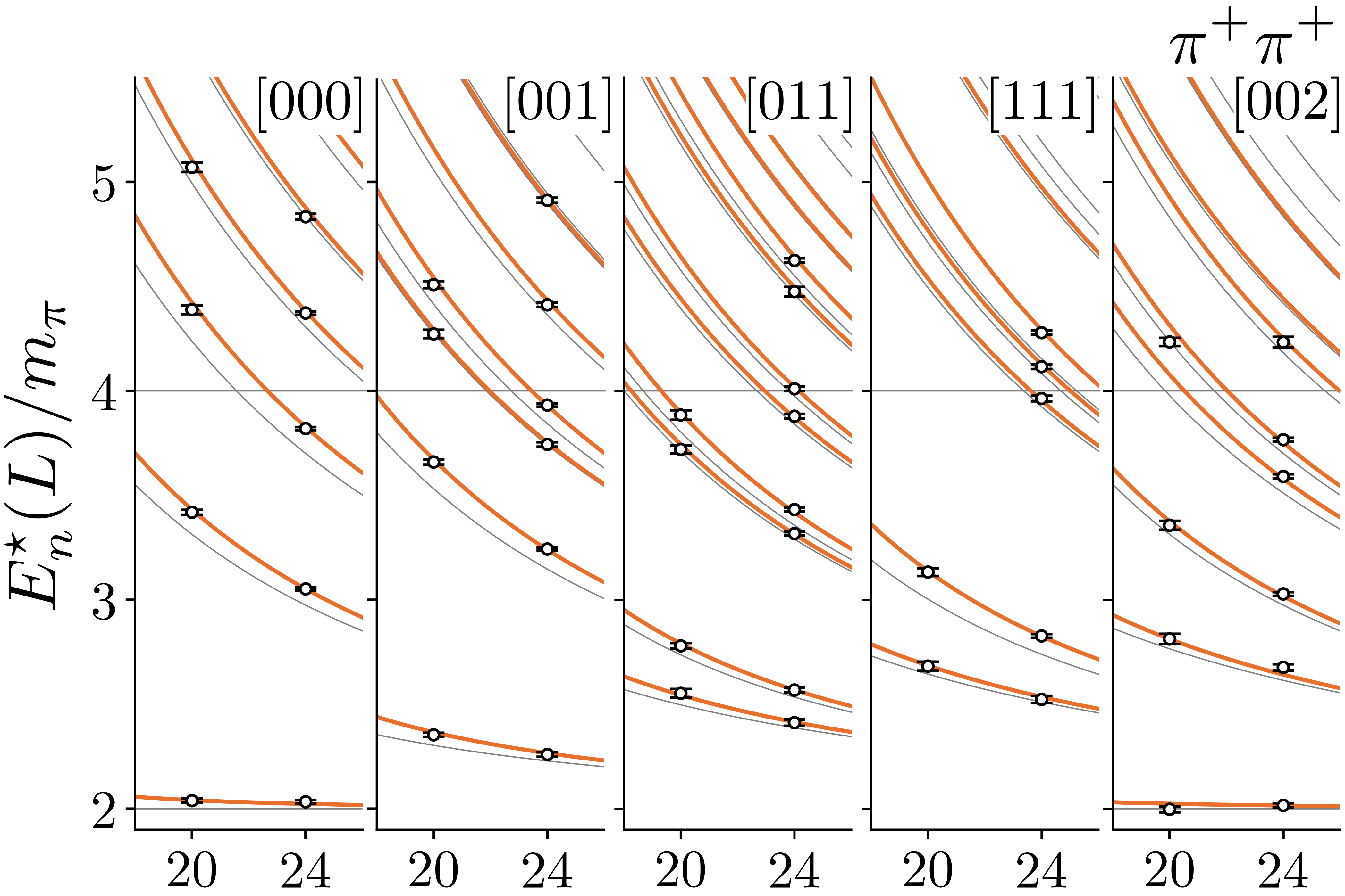
$I = 3\,(\pi^+\pi^+\pi^+),\ \boldsymbol{P} = [000],\ \Lambda = A_1^-,\ L/a_s = 24$



Good operators
+ distillation!

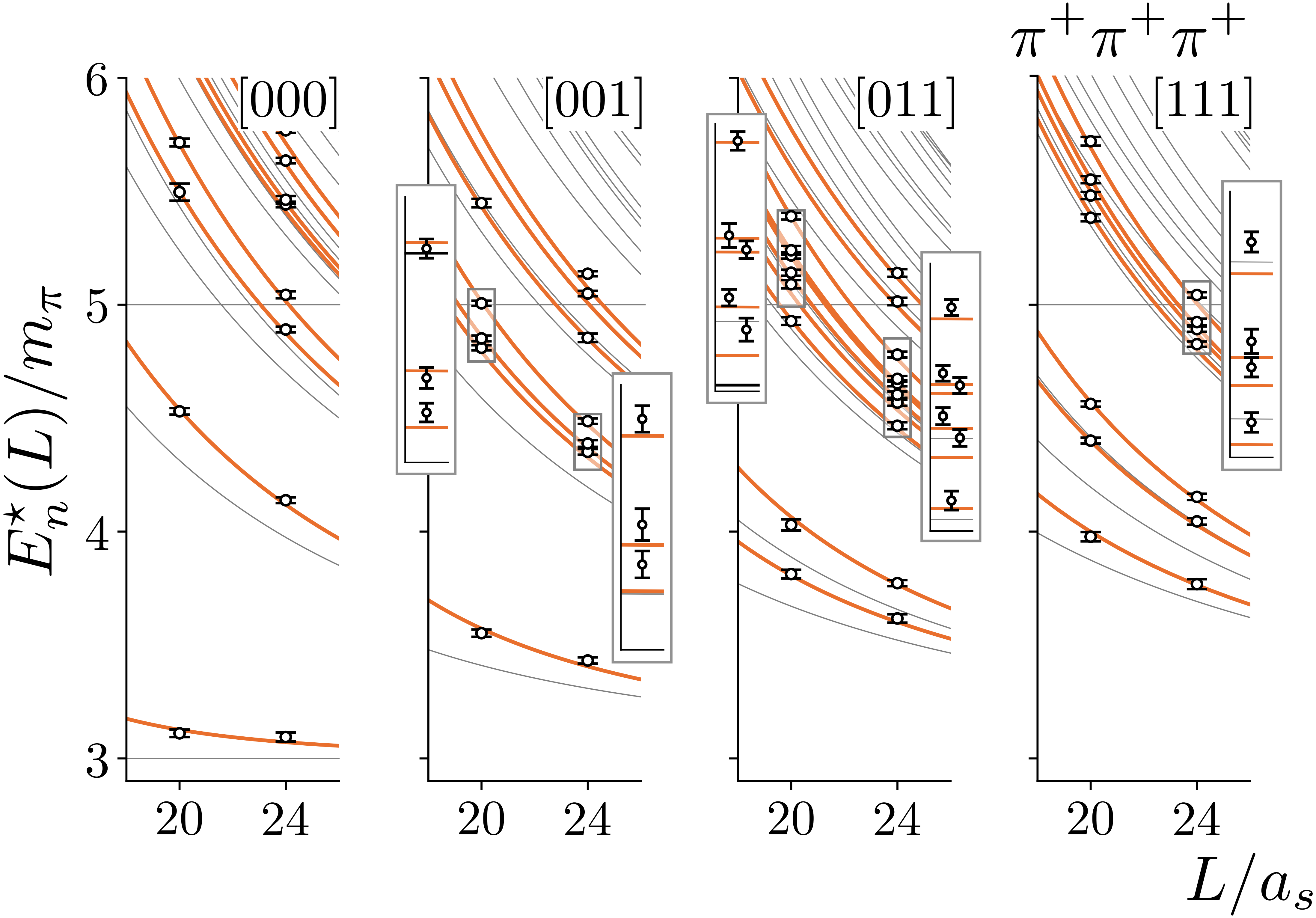
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+\pi^+$ energies



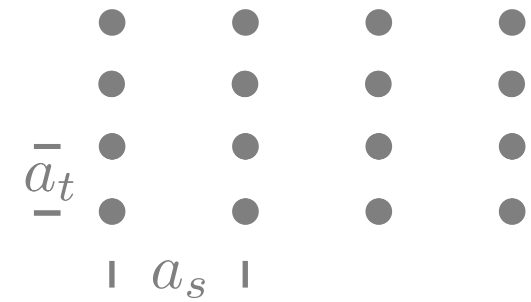
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

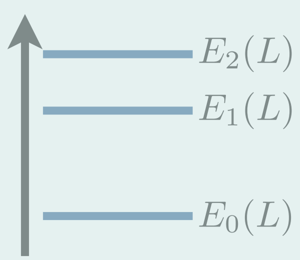
lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$ $\quad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$



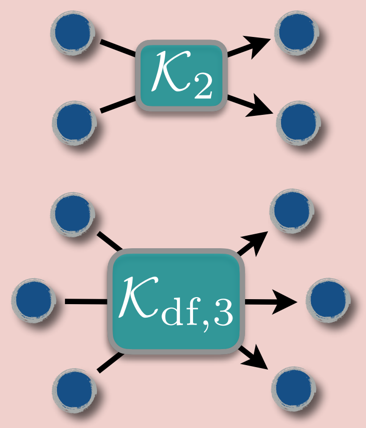
☐ Workflow outline



Determine finite-volume energy spectrum

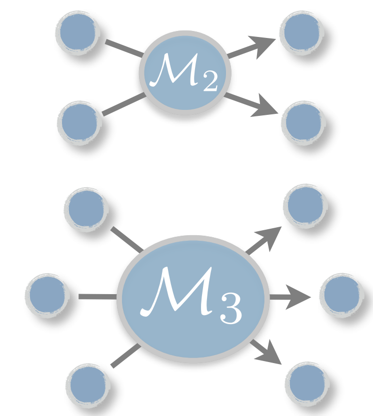
✓

finite volume ⟷



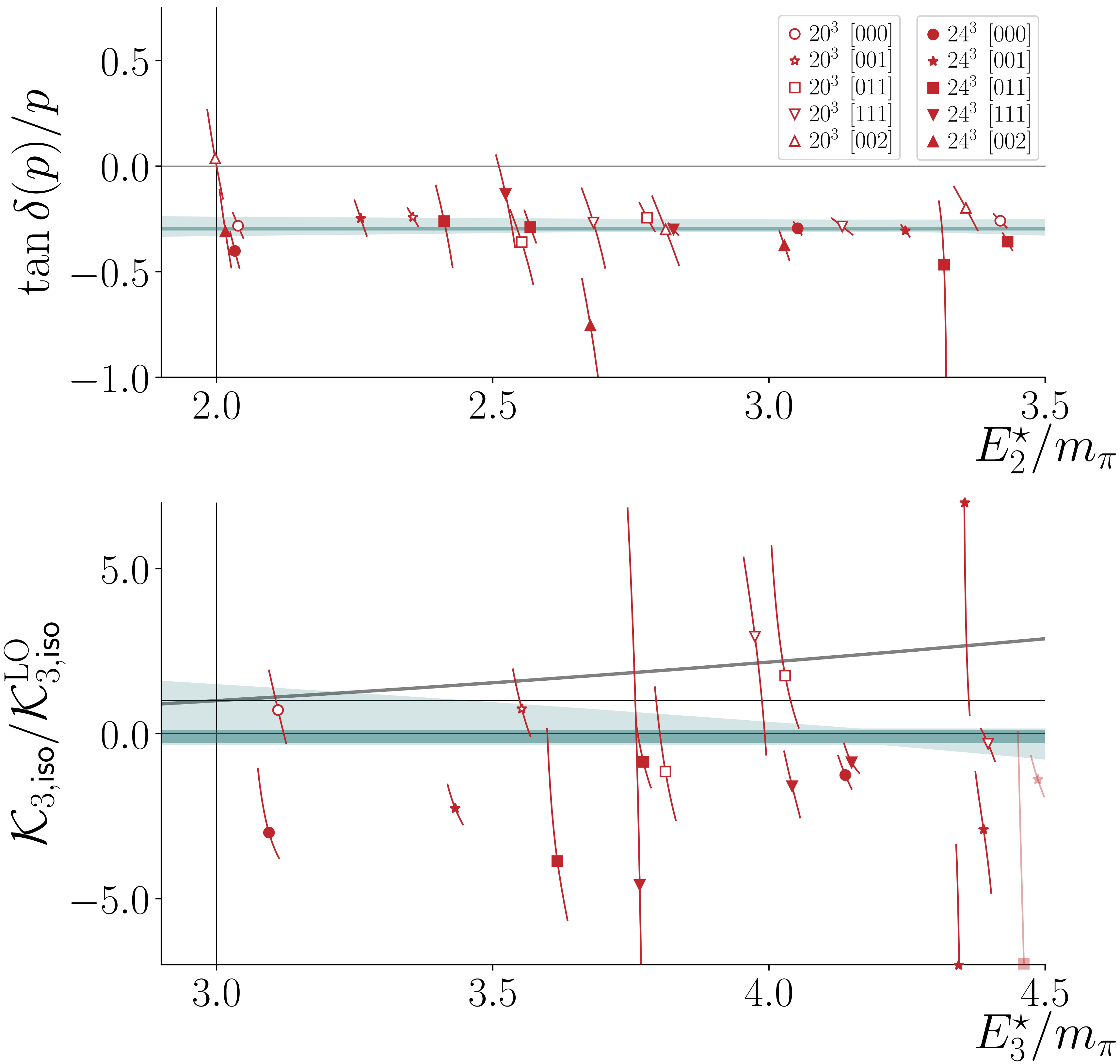
Fit to constrain two- and three-body K matrices

unitarity ⟷



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

K matrix fits



Finite-volume formalism relates energies to K matrices

One-to-one for $K_{\text{df},3}$ depending only on $E_{\text{cm}} = E^\star$

Fit both two and three-body K to various polynomials

Cut on the CM energy in the fits

$K_{\text{df},3}$ is scheme dependent (removed upon converting to $\mathcal{M}_3$)

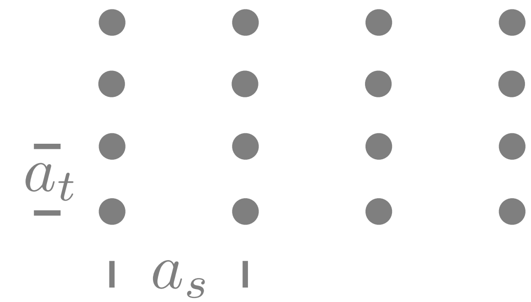
$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

lattice details

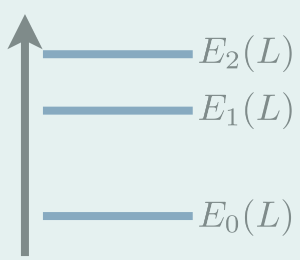
$N_f = 2+1 \qquad a_s/a_t = 3.444(6)$

$m_\pi \approx 400\text{MeV} \qquad a_s \approx 0.12\text{fm}$

$L_s/a_s = 20, 24$



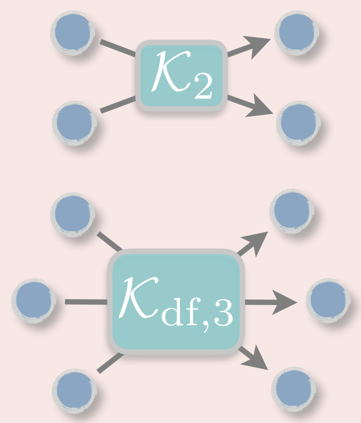
Workflow outline



Determine finite-volume energy spectrum

✓

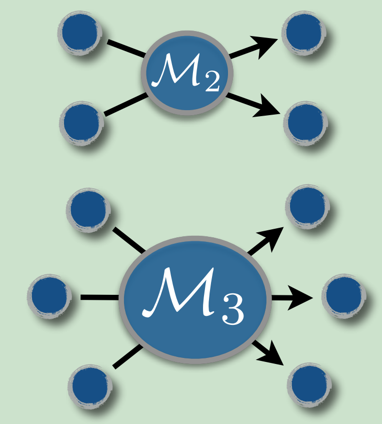
finite volume ⟷



Fit to constrain two- and three-body K matrices

✓

unitarity ⟷

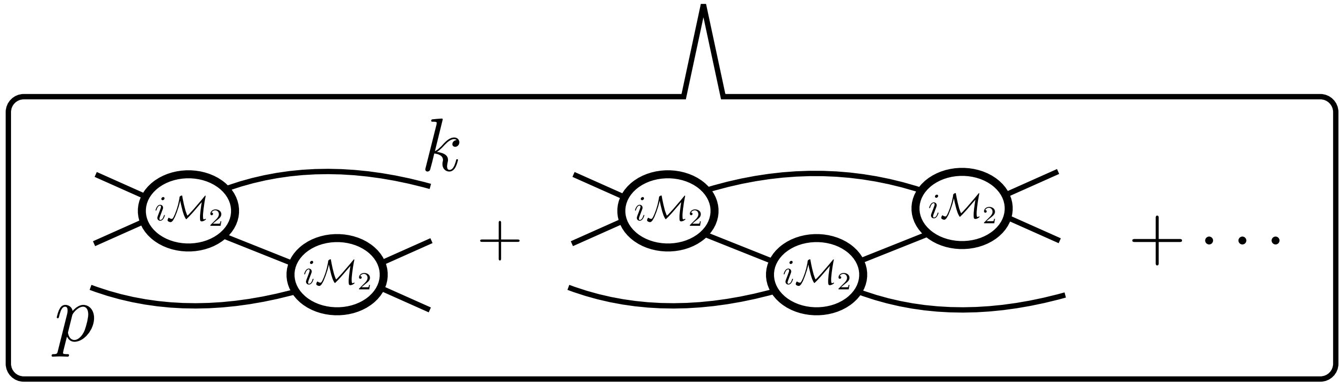


Solve integral equations to extract $3 \to 3$ scattering

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

Integral equation

$$\mathcal{M}_3^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) + \mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{p})\mathcal{T}(E_3^\star)\mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{k})$$

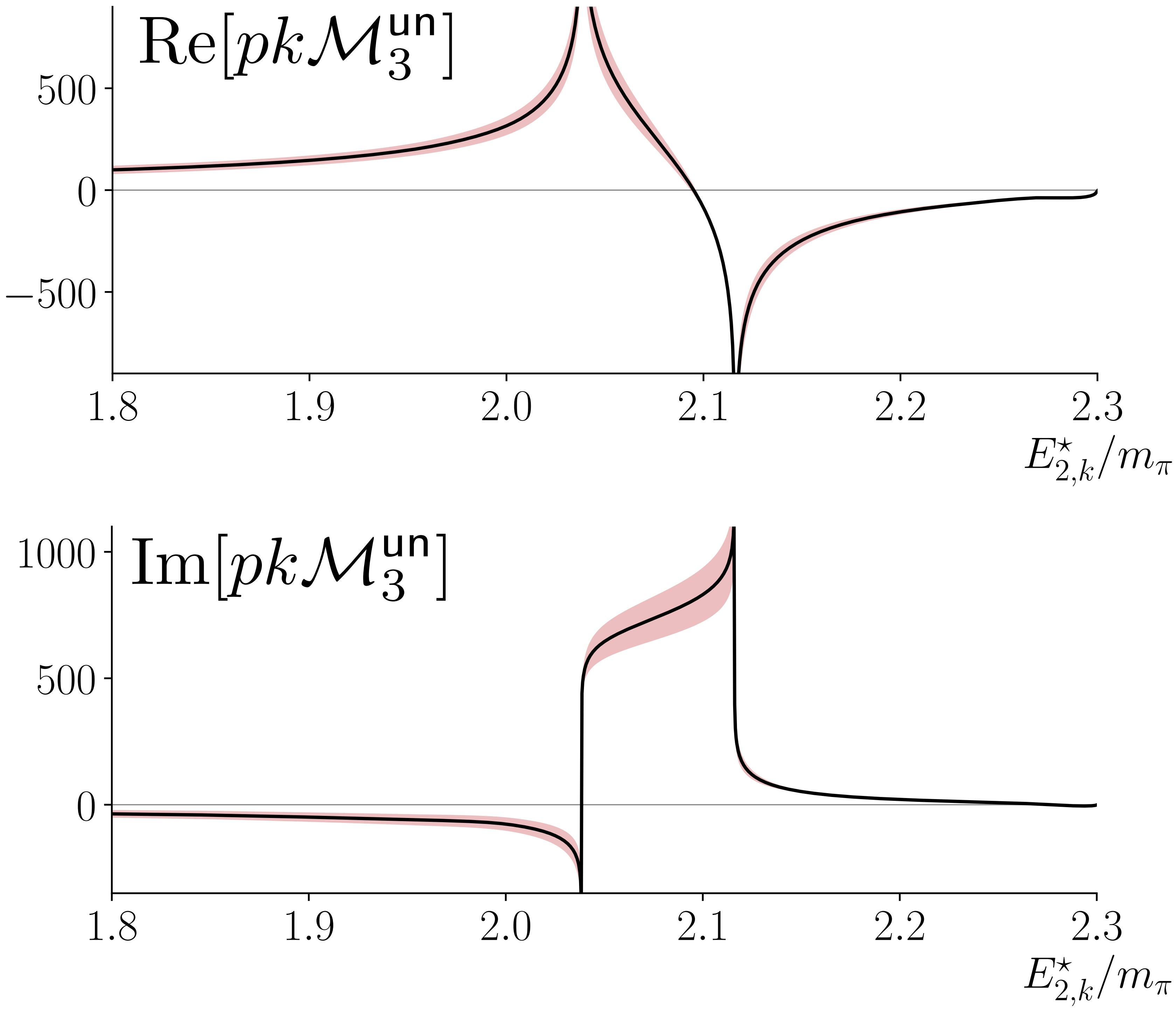


Vanishes for $K_{\mathrm{df},3} = 0$

$$\boldsymbol{D}(N, \epsilon) = -\boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{P} \cdot \boldsymbol{D}(N, \epsilon)$$

$$\mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \lim_{\epsilon \to 0} \lim_{N \to \infty} \boldsymbol{D}_{pk}(N, \epsilon)$$

Integral equation



Total angular momentum = 0

Two-particle sub-system angular momentum = 0

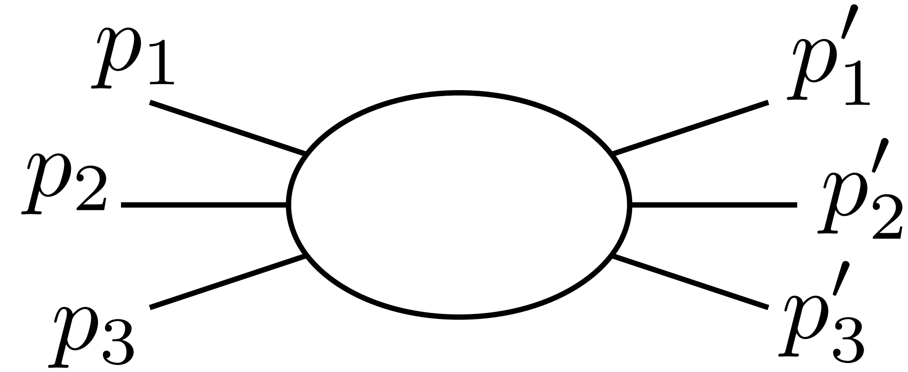
Plot at fixed $E_3^\star$ and p

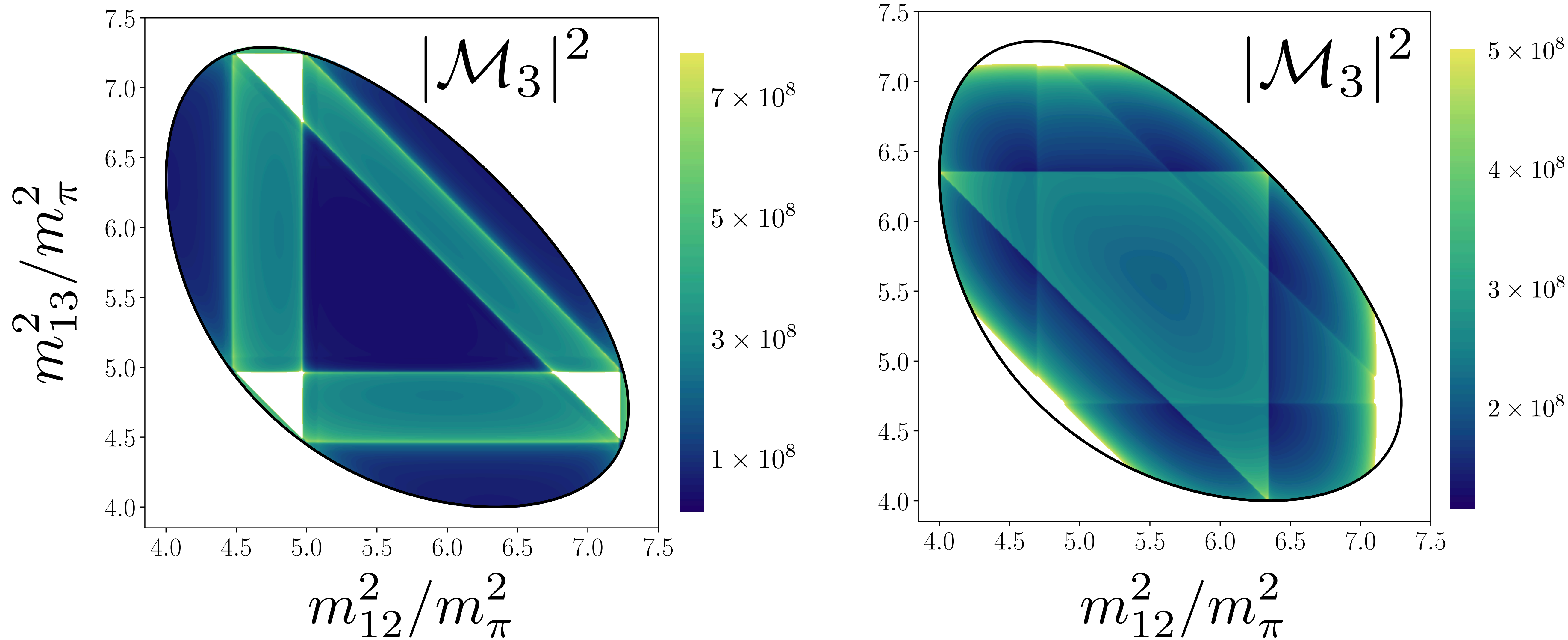
Both two- and three-body uncertainties estimated

Still need to symmetrize

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\mathcal{M}_3 = \sum_{i,j \in \{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p_i', p_j)$$





MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

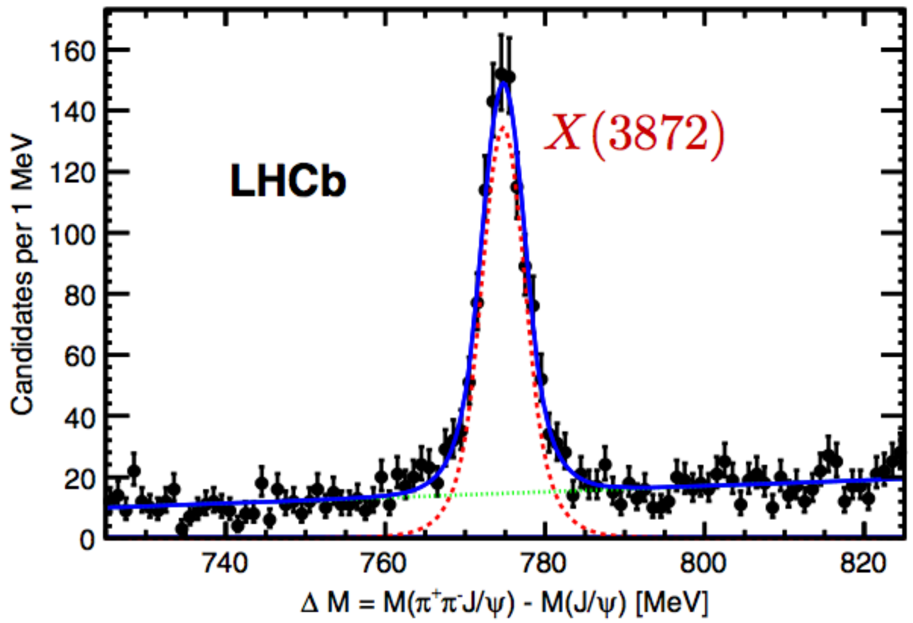
Outlook for exotics

- Challenging for many reasons

incredible hierarchy of scales

$$M_X - M_{D^0 D^{0*}} = 0.01 \pm 0.18\,\text{MeV}$$

$$\Gamma(X) < 1.2\text{MeV}$$



- Lebed, Mitchell, Swanson, *PPNP review* (2017)
- LHCb (PRD92, 2015)

many open channels $X \to \omega J/\psi,\ \pi\pi J/\psi,\ D^{*0}\bar{D}^0,\ \cdots$

- My speculative outlook

should be possible to derive N-particle volume formalism

then relate K-matrices to spectra to identify LQCD strategy

hierarchy of scales = not always a problem $\mathcal{K}_2 \propto a \to \infty$

Five-day outline

- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} volume effects
  - ☑ Mass, matrix element in $\lambda\phi^4, g\phi^3$
  - ☑ Pion mass
  - ☑ LO-HVP for $(g-2)_\mu$
  - ☑ Bethe-Salpeter kernel
- ☑ $2 \to 2$ finite-volume (f.v.) formalism
  - ☑ Scattering basics
  - ☑ Derivation
  - ☑ Generalizations
  - ☑ Applications
- ☑ $(1+)\mathscr{J} \to 2$ f.v. formalism
  - ☑ Derivation
  - ☑ Expansions
  - ☑ Applications
- ☑ $2 + \mathscr{J} \to 2$ f.v. formalism
  - ☑ Derivation
  - ☑ Testing the result
  - ☑ Numerical explorations
- ☑ Non-local matrix elements
  - ☑ Derivation
  - ☑ Applications
- ☑ $3 \to 3$ f.v. formalism
  - ☑ New complications
  - ☑ Derivation ($E_n(L)$ to $\mathscr{K}_{\mathrm{df},3}$)
  - ☑ Integral equations ($\mathscr{K}_{\mathrm{df},3}$ to $\mathscr{M}_3$)
  - ☑ Testing the result
  - ☑ Numerical explorations/calculations
- ☐ Finite-volume QED + QCD
  - ☐ Different formalisms
  - ☐ Basic examples