UK Research and Innovation

QCD (and QED) in a Euclidean finite volume

EuroPLEx Summer School 2021

Maxwell T. Hansen

August 23rd-27th

THE UNIVERSITY of EDINBURGH

Five-day outline

- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} volume effects
  - ☑ Mass, matrix element in $\lambda\phi^4, g\phi^3$
  - ☑ Pion mass
  - ☑ LO-HVP for $(g-2)_\mu$
  - ☑ Bethe-Salpeter kernel
- ☑ $2 \to 2$ finite-volume (f.v.) formalism
  - ☑ Scattering basics
  - ☑ Derivation
  - ☑ Generalizations
  - ☑ Applications
- ☑ $(1+)\mathscr{J} \to 2$ f.v. formalism
  - ☑ Derivation
  - ☑ Expansions
  - ☑ Applications
- ☑ $2 + \mathscr{J} \to 2$ f.v. formalism
  - ☑ Derivation
  - ☑ Testing the result
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  - ☑ Derivation ($E_n(L)$ to $\mathscr{K}_{\mathrm{df},3}$)
  - ☑ Integral equations ($\mathscr{K}_{\mathrm{df},3}$ to $\mathscr{M}_3$)
  - ☑ Testing the result
  - ☑ Numerical explorations/calculations
- ☐ Finite-volume QED + QCD
  - ☐ Different formalisms
  - ☐ Basic examples

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Why QED + QCD?

- In our Universe, up and down quarks have different masses and electric charges
- Thus far we have been ignoring these effects

$$\langle \mathcal{O} \rangle_{[1+1+1,\ \alpha_{\mathsf{QED}}=1/137]} \approx \langle \mathcal{O} \rangle_{[2+1,\ \alpha_{\mathsf{QED}}=0]}$$

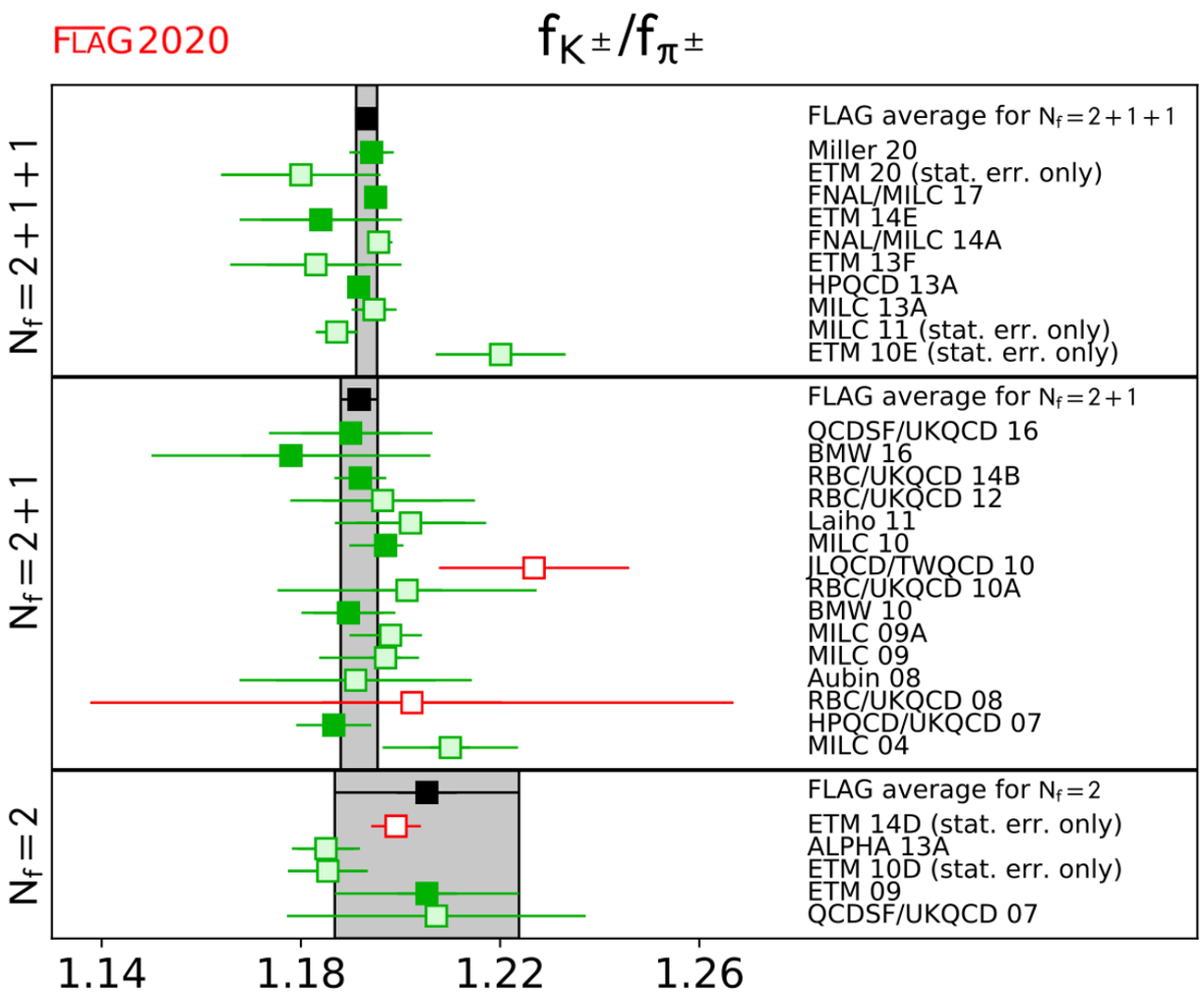
- This is expected to induce a percent-level systematic uncertainty

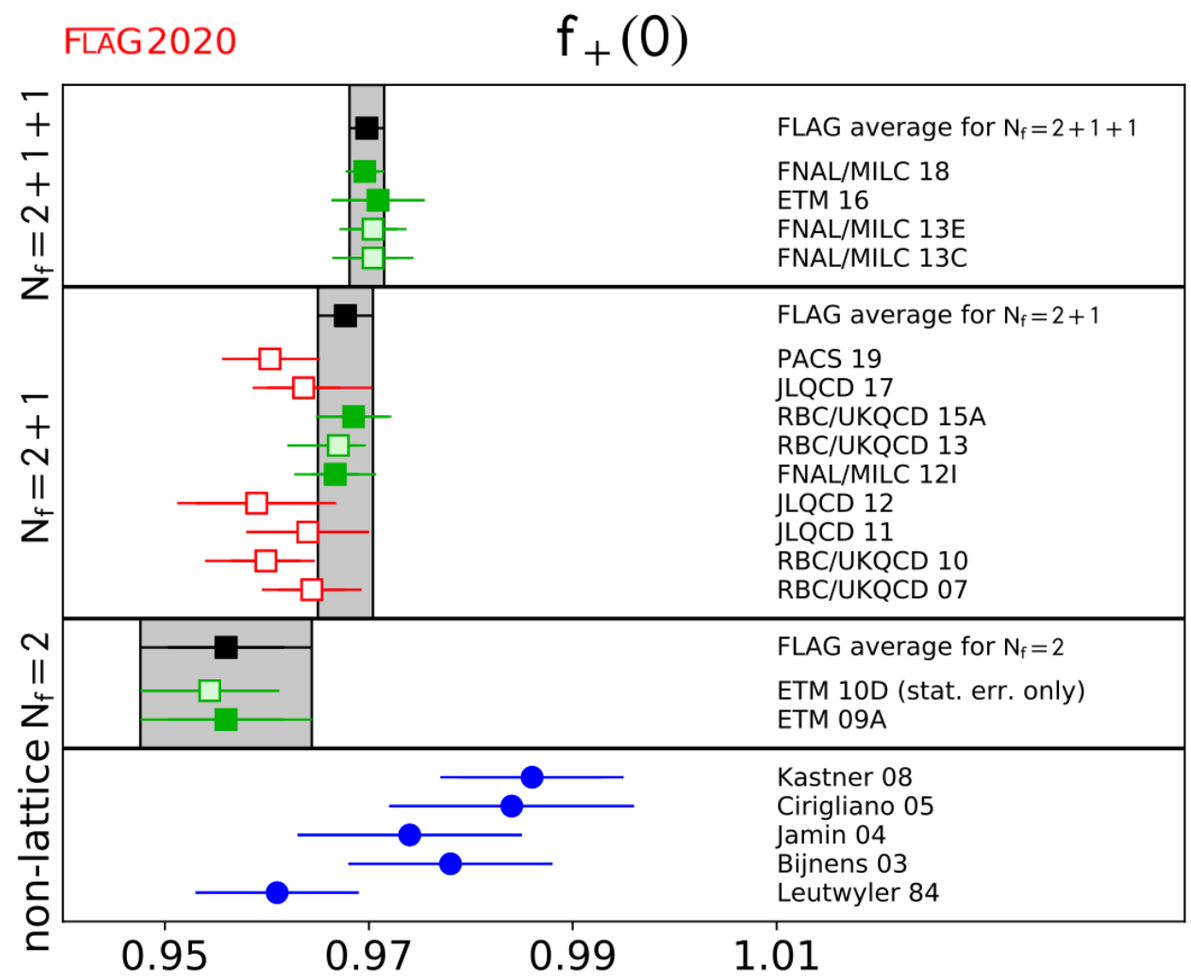
$$\alpha_{\mathsf{QED}} \approx 0.7\,\% \qquad \frac{M_n - M_p}{M_n} \approx 0.1\%$$

- But many LQCD observables have reached percent level determinations!

Precision era LQCD

$f_\pi = 130.2(8)$ MeV	uncertainty = 0.6%	$\delta_{\chi PT,\ \mathrm{QED}}(\pi^- \to \ell^- \bar{\nu}) = 1.8\,\%$
$f_K = 155.7(0.7)$ MeV	uncertainty = 0.5%	$\delta_{\chi PT,\ \mathrm{QED}}(K^- \to \ell^- \bar{\nu}) = 1.1\,\%$
$f_+(0) = 0.9698(17)$	uncertainty = 0.2%	$\delta_{\chi PT,\ \mathrm{QED}}(K \to \pi \ell \bar{\nu}) = 0.5$ to $3\,\%$



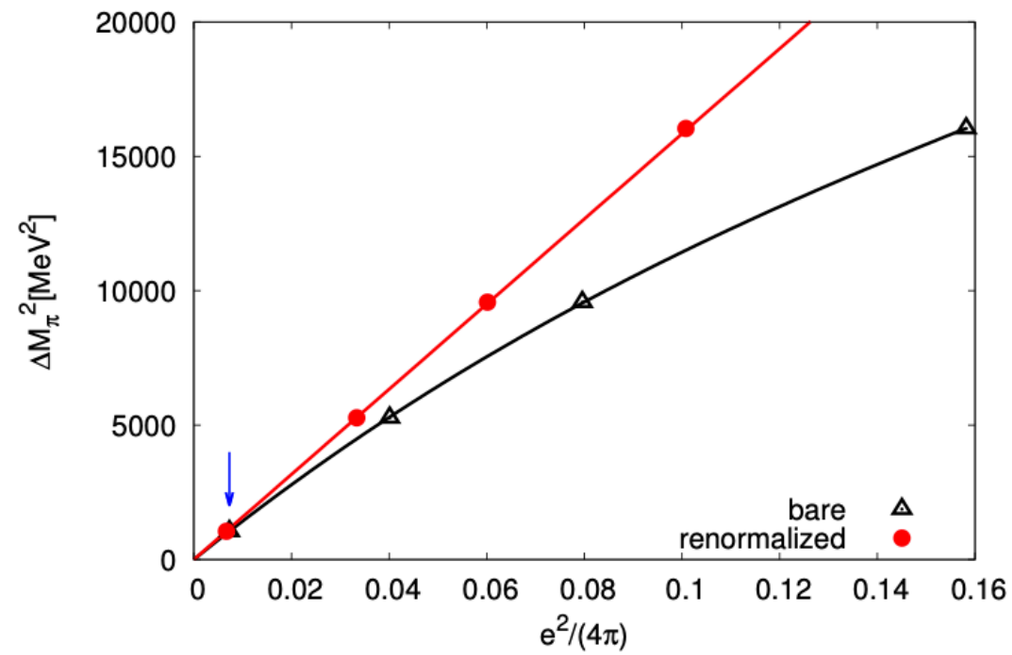


Two basic strategies for QED+QCD

- Simulate with α_{QED}, $m_u - m_d$

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\Phi \, \mathcal{O} \, e^{-S_{\text{QCD}} - S_{\text{QED+IB}}}$$

simplifies observables, but signal may be suppressed



Borsanyi et al., Science 347 (2015) 1452-1455

- Expand in α_{QED}, $m_u - m_d$

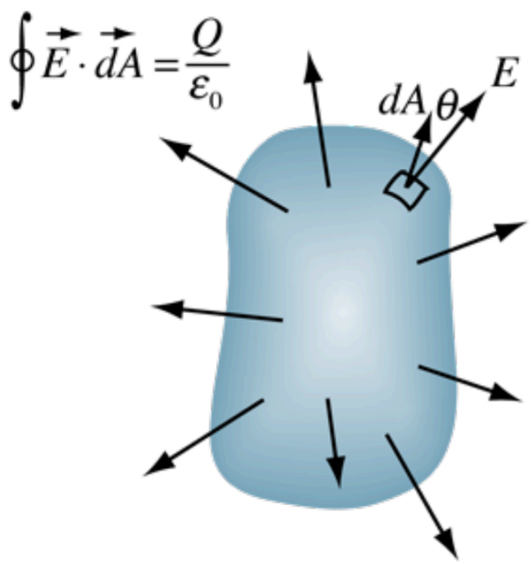
$$\langle \mathcal{O} \rangle = \int \mathcal{D}\Phi \, \mathcal{O} \, e^{-S_{\text{QCD}}} - \int \mathcal{D}\Phi \left[S_{\text{QED+IB}} \mathcal{O} \right] e^{-S_{\text{QCD}}} + \mathcal{O}(e^4)$$

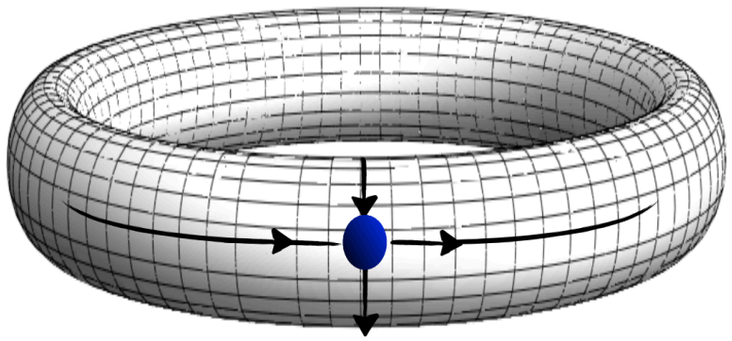
signal is not suppressed, but observables are more complicated

QED in a box

Gauss's law prohibits a naive implementation of charged objects in a periodic volume

$$Q = \int d^3\boldsymbol{x}\, j_0(t,\boldsymbol{x}) = \int d^3\boldsymbol{x}\, \nabla\cdot\boldsymbol{E}(t,\boldsymbol{x}) = \int d\boldsymbol{S}\cdot\boldsymbol{E}(t,\boldsymbol{x}) = 0$$





Note: if expanding one can take infinite-volume QED

but putting QED and QCD in different spaces causes its own subtleties

- figures from A. Nicholson, GHP 2021 •

Many proposed methods

- Remove the global zero-mode of the gauge field (QED_{TL})
- Restrict the global zero-mode of the gauge field
- Remove the spatial zero-mode of the gauge field in each timeslice (QED_L)
- Massive photon (QED_M)
- $C^\star$ boundary conditions (QED_C)

All equivalent if $L \to \infty$ before anything else
(before $a \to 0$, before $T \to \infty$, before $m_\gamma \to 0$, maybe before fitting)

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Don't mess with the global zero mode!

- Remove the global zero-mode of the gauge field (QED_{TL})

$$a_\mu = eL_\mu \int d^4x A_\mu(x) \stackrel{!}{=} 0$$

- Restrict the global zero-mode of the gauge field

$$-\pi \;<\; a_\mu \;<\; \pi$$

Both of these disrupt the transfer matrix! (i.e. the hamiltonian)

$$\int d^3\boldsymbol{x} \langle \psi(t,\boldsymbol{x})\overline{\psi}(0)\rangle \neq \sum_{n,m} C_{nm}(L) e^{-t(E_n - E_m)} e^{-TE_m}$$

Must send $L \to \infty$ before making use of spectral decomposition

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$\mathrm{QED_L}$

$$\int d^3\boldsymbol{x}\, A_\mu(x) \stackrel{!}{=} 0$$

$\mathrm{QED_L}$ has a transfer matrix. It is a nonlocal prescription. Locality is a core property of QFT, it is a fundamental assumption behind

- Renormalizability by power counting
- Volume-independence of renormalization constants
- Operator product expansion
- Effective-theory description of long-distance physics
- Symanzik improvement program
- ...

Infinite-volume limit should be taken before the continuum limit.

$\mathrm{QED_L}$ has a transfer matrix

$\mathrm{QED_L}$ is non-local

- slide from A. Patella, Lattice 2016 plenary (not the meme though) •

$\mathrm{QED_L}$

$$\int d^3\boldsymbol{x}\, A_\mu(x) \stackrel{!}{=} 0$$

- In matrix elements of high-dimension operators one expects terms scaling as

$$\frac{1}{aL^n}$$

Vanishes if $L \to \infty$ at fixed a,
diverges if $a \to 0$ at fixed L

- No specific evidence of a problem for quantities currently being calculated
- Failure of NREFT = important lesson
- Would be very interesting to improve understanding of these issues

Practical consequence of QED_L = modified Feynman rules in calculating volume effects

$$\frac{1}{L^3}\sum_{\boldsymbol{k}} \longrightarrow \frac{1}{L^3}\sum_{\boldsymbol{k}\neq\boldsymbol{0}}$$

Has anyone looked at exercise **3**?

3.1 Evaluating the Bessel integral

In lecture we found the mass shift for $\lambda\phi^4$ theory in a periodic cubic volume:

$$\Delta m^2(L) \equiv m(L)^2 - m^2 = \frac{\lambda_0}{2}\Delta_{\boldsymbol{k}} \int \frac{dk_4}{2\pi} \frac{1}{k^2 + m^2} \,,$$

where $k^2 = k_4^2 + \boldsymbol{k}^2$ and

$$\Delta_{\boldsymbol{k}} \equiv \left[\frac{1}{L^3} \sum_{\boldsymbol{k}} - \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \right] .$$

3.2 Zero-mode quenching

Suppose we modify the theory such that

$$\Delta_{\boldsymbol{k}} \longrightarrow \Delta'_{\boldsymbol{k}} \equiv \left[\frac{1}{L^3} \sum_{\boldsymbol{k}\neq\boldsymbol{0}} - \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \right] .$$

Give the full result for the $\mathcal{O}(\lambda)$ value of $\Delta m^2(L)$ in this case.

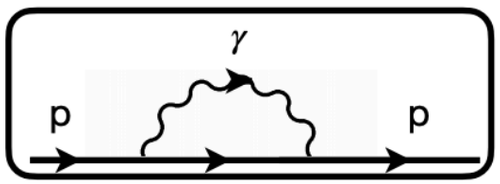
- This is really $(g\phi^3)_L$

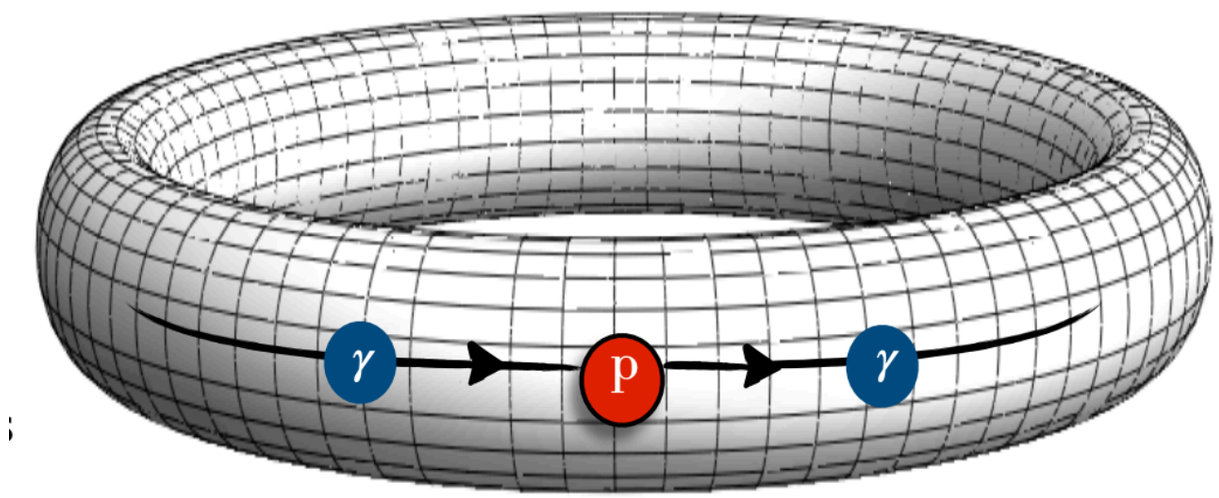
$$\Delta m(L) = \mathcal{O}(1/L^3)$$

$\mathrm{QED_L}$ mass determination

$\mathrm{QED_M}$

- Combine Landau gauge + mass term for the photon
- Gauge invariance is broken, but in a controlled way





$$\delta_{\mathrm{FV}} \sim e^{-m_\gamma L}$$

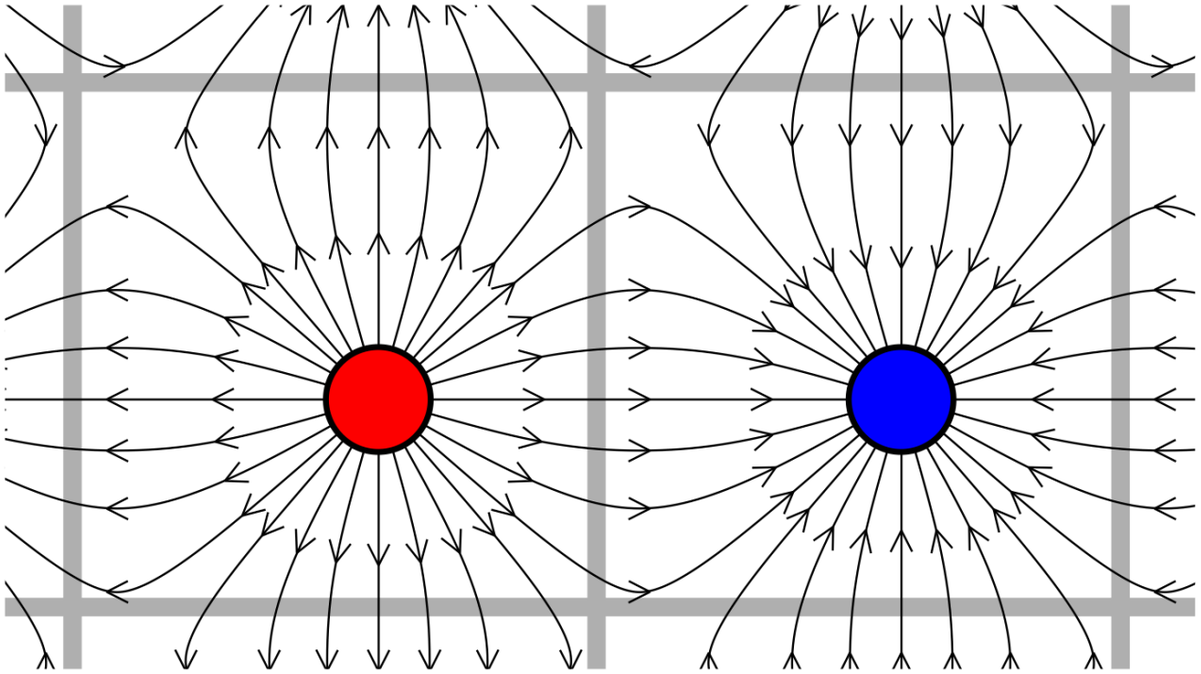
- Endres et al., arXiv:1507.08916, (2015) •

$\mathrm{QED}_{\mathrm{C}}$

- Use charge-conjugation-like boundary conditions

$$A_\mu(x + L\hat{\boldsymbol{e}}_i) = -A^*_\mu(x)$$

$$\psi(x + L\hat{\boldsymbol{e}}_i) = C^{-1}\overline{\psi}^T(x)$$



- Still power-like volume effects, but leading non-universal term is removed

- Exponentially suppressed flavor mixing

• Wiese (1992) • Polley (1993) • Lucini *et al.* (2016) •

Not discussed here…

- Scheme dependence of separating α_{QED} from $(m_u - m_d)$ (see e.g. FLAG discussion)
- Observables beyond masses

QCD + QED captures why lattice field theory is so fantastic!

Methods, algorithms, computing improves

Demand new effects to be considered

Demand new methods and algorithms, drive next generation computing

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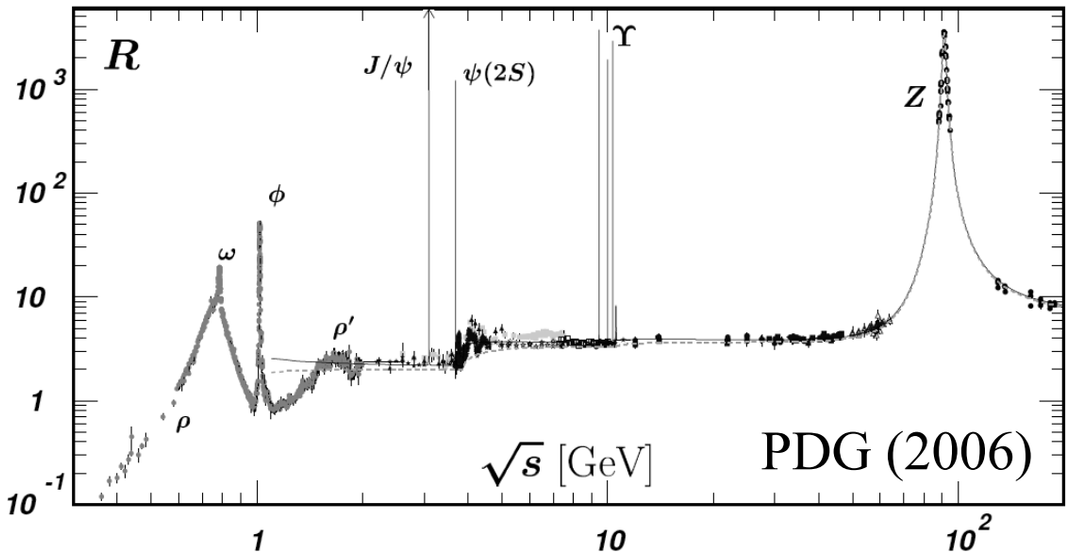
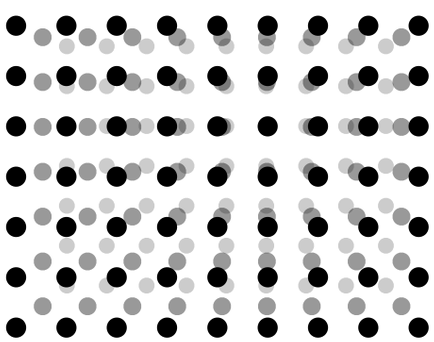
Invitation

- R-ratio based (g-2) determination

$$a^{\mathrm{HVP,LO}}(m_\mu) = \int_{4M_\pi^2}^{\infty} ds\, K(m_\mu, s)\, R(s)$$



- Integral can be Wick rotated to Euclidean signature… but ignore this



$$\longrightarrow \quad G(\tau) = \int_{4M_\pi^2}^{\infty} ds\, g(\tau, s)\, R(s) \qquad g(\tau, s) \propto \sqrt{s} e^{-\sqrt{s}\tau}$$

- Can we find a function $\mathcal{K}(m_\mu, \tau)$? such that…

$$\int_0^\infty d\tau\, \mathcal{K}(m_\mu, \tau)\, g(\tau, s) = K(m_\mu, s) \quad \longrightarrow \quad a^{\mathrm{HVP,LO}}(m_\mu) = \int_0^\infty d\tau\, \mathcal{K}(m_\mu, \tau)\, G(\tau)$$

Of course this defines the standard kernel

General idea

we want…

$$\widehat{\rho}(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho(\omega)$$

we have…

$$G(\tau) = \int d\omega\, e^{-\omega\tau}\, \rho(\omega)$$

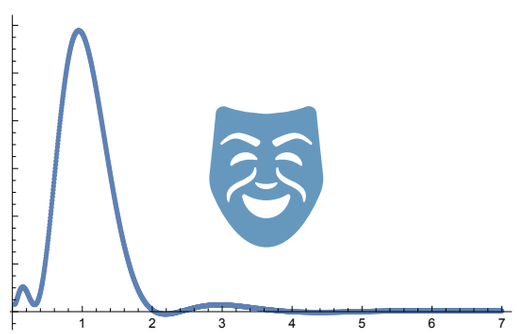
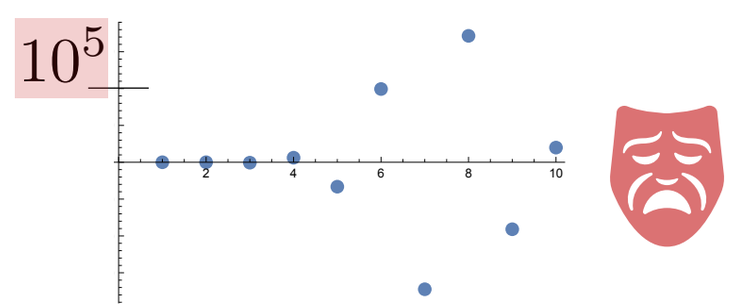
we try to construct

$$\int_0^\infty d\tau\, \mathcal{K}(\bar{\omega},\tau)\, e^{-\omega\tau} = \widehat{\delta}_\Delta(\bar{\omega},\omega)$$

it then follows that

$$\widehat{\rho}(\bar{\omega}) = \int_0^\infty d\tau\, \mathcal{K}(\bar{\omega},\tau)\, G(\tau)$$

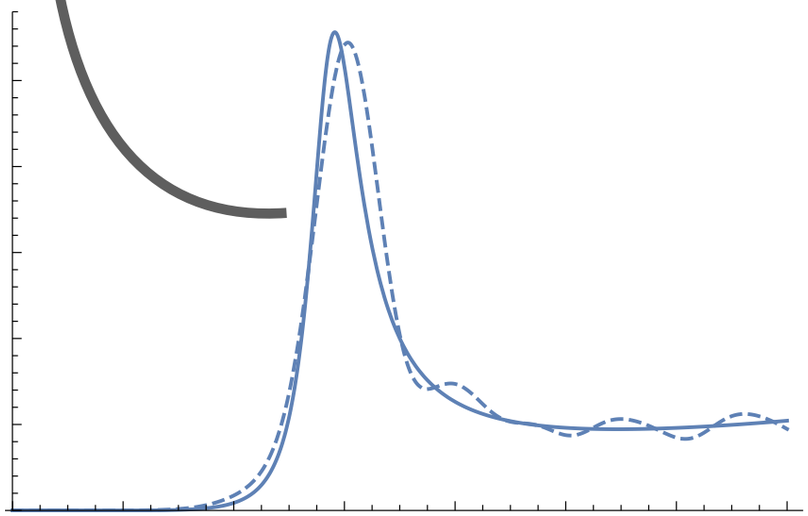
some examples…

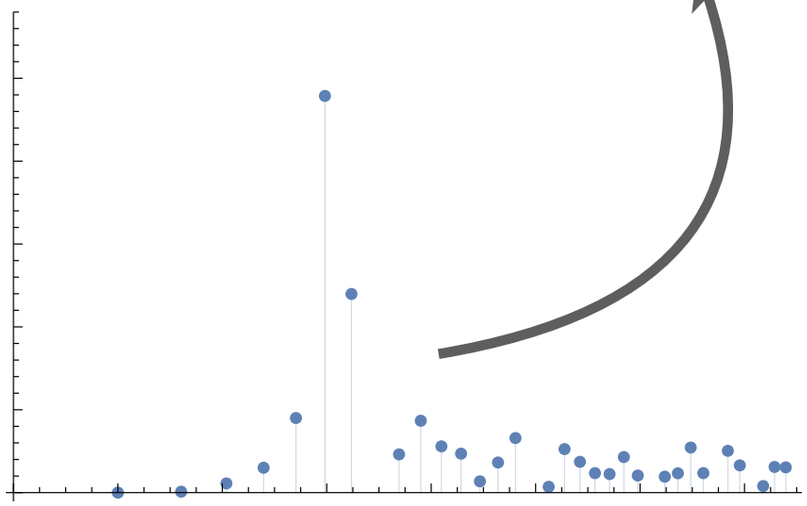
$\widehat{\delta}_\Delta(\bar{\omega},\omega)$	$\mathcal{K}(\bar{\omega},\tau)$
$e^{-\omega/\bar{\omega}}$	$\delta(\bar{\omega} - 1/\tau)$
$\dfrac{1}{\bar{\omega}-\omega}$	$e^{\bar{\omega}\tau}$
	

Role of the finite volume

$$\widehat{\rho}_L(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho_L(\omega)$$

$$G_L(\tau) = \int d\omega\, e^{-\omega\tau}\, \rho_L(\omega)$$





- Any reconstructed spectral function that ≠ forest of deltas…

contains implicit smearing (or else $L \to \infty$)

We require…

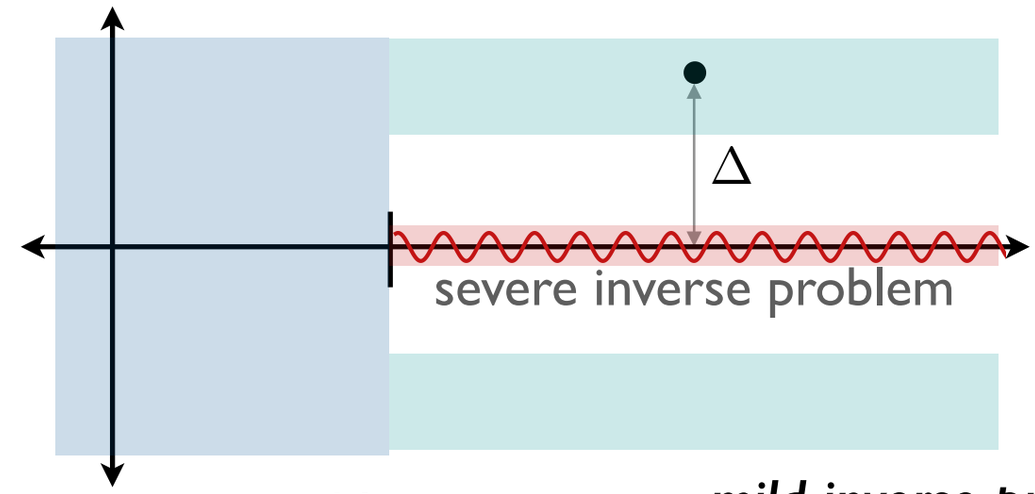
$$1/L \ll \Delta \ll \mu_{\text{physical}}$$

smearing function covers many delta peaks

smearing does not overly distort observable

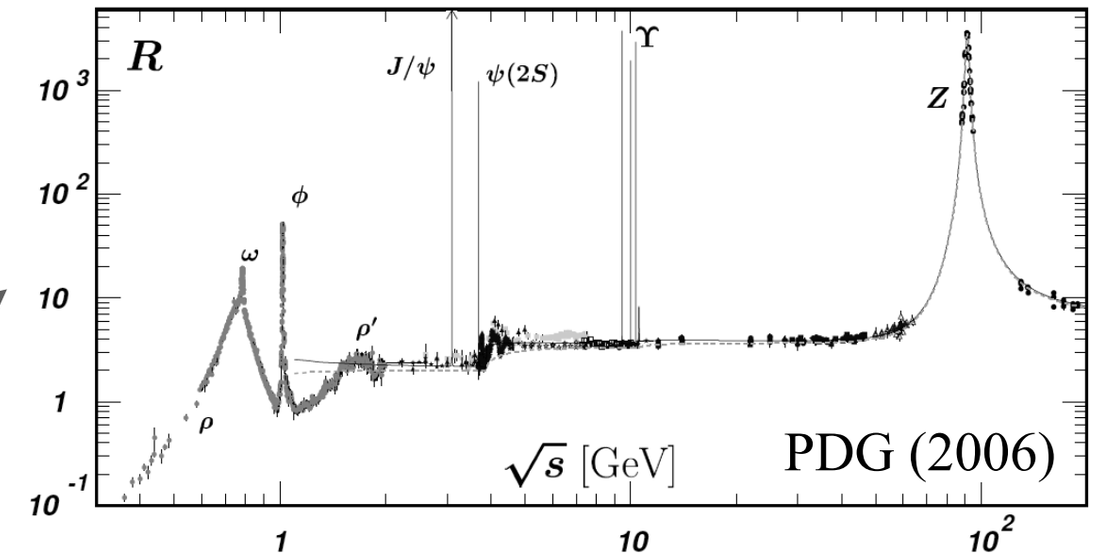
MTH, Meyer, Robaina (2017)

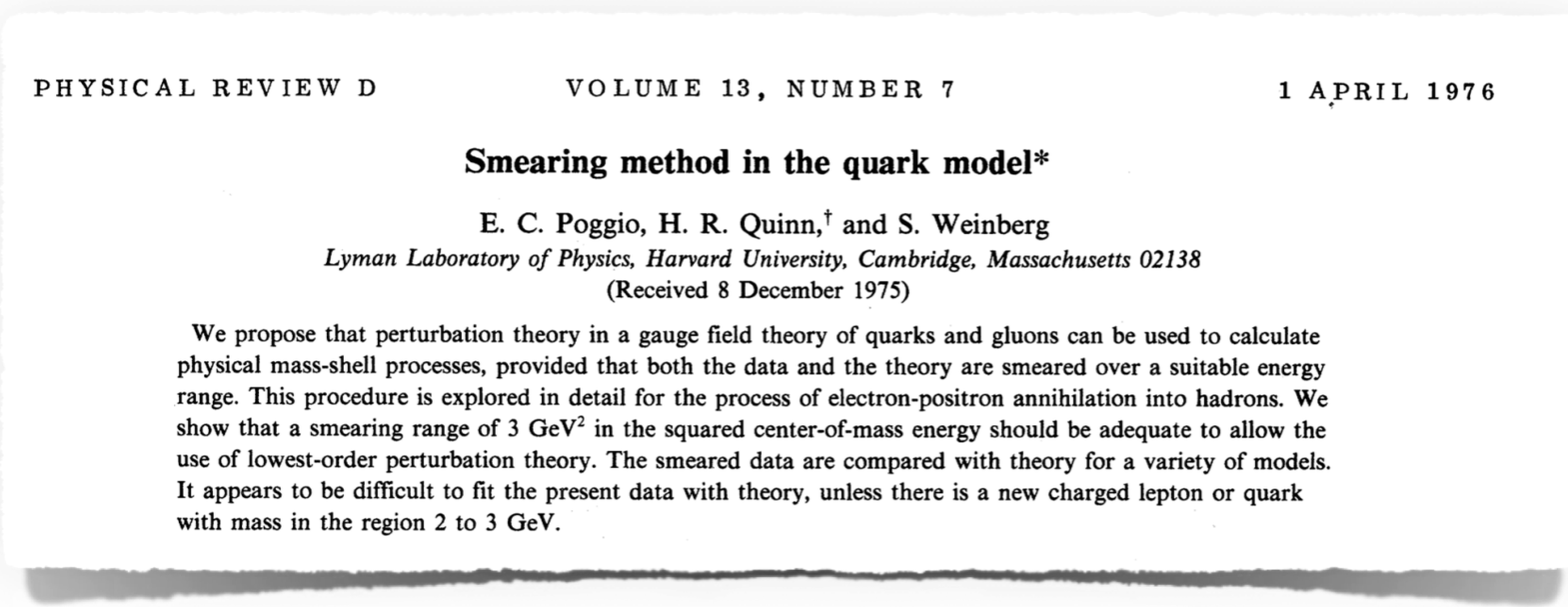
R ratio



Hashimoto

mild inverse problem?





PHYSICAL REVIEW D VOLUME 13, NUMBER 7 1 APRIL 1976

Smearing method in the quark model*

E. C. Poggio, H. R. Quinn,† and S. Weinberg

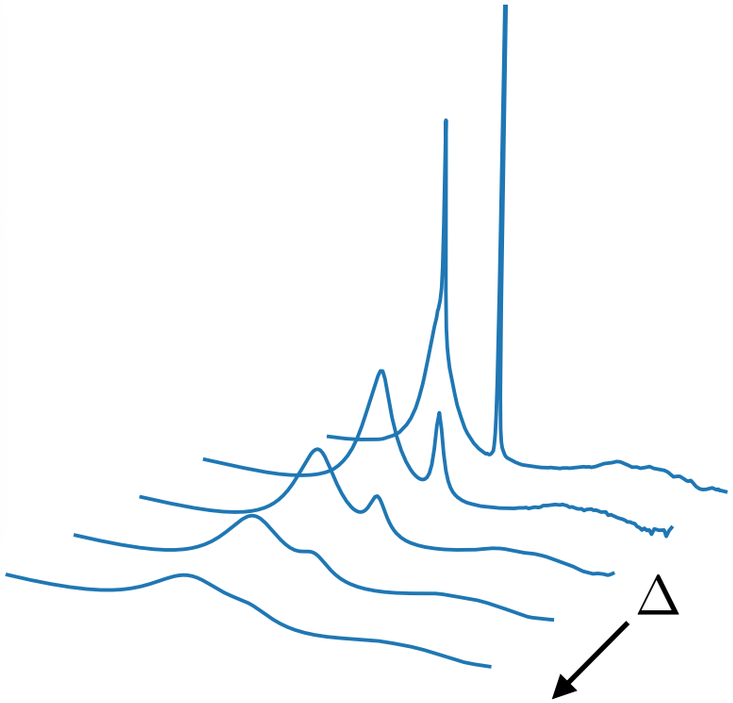
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 8 December 1975)

We propose that perturbation theory in a gauge field theory of quarks and gluons can be used to calculate physical mass-shell processes, provided that both the data and the theory are smeared over a suitable energy range. This procedure is explored in detail for the process of electron-positron annihilation into hadrons. We show that a smearing range of 3 GeV^2 in the squared center-of-mass energy should be adequate to allow the use of lowest-order perturbation theory. The smeared data are compared with theory for a variety of models. It appears to be difficult to fit the present data with theory, unless there is a new charged lepton or quark with mass in the region 2 to 3 GeV.

$$\widehat{R}_\Delta(s) = \frac{\Delta}{\pi}\int_0^\infty ds' \, \frac{R(s')}{(s'-s)^2+\Delta^2}$$

into the complex plane!



courtesy of M. Bruno
(using F. Jegerlehner's ***alphaQED***)

Backus-Gilbert

- Developed by geophysicists Backus and Gilbert in 1967

The Resolving Power of Gross Earth Data

George Backus and Freeman Gilbert

(Received 1968 February 12)

- ***Linear, model-independent*** → smeared solution with a ***known resolution function***

$$\sum_\tau \mathcal{K}(\bar{\omega},\tau)G(\tau) = \sum_\tau \mathcal{K}(\bar{\omega},\tau)\int d\omega\, e^{-\omega\tau}\,\rho(\omega)$$

BG optimizes *K*

$$= \int d\omega \left[\sum_\tau \mathcal{K}(\bar{\omega},\tau)\, e^{-\omega\tau}\right]\rho(\omega)$$

$$= \int d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\,\rho(\omega)$$

δ is exactly known

- The correlation matrix is used to stabilize the inverse

- The quality of the data determines the ***resolution***

Backus-Gilbert (details)

- Un-stabilized inverse

Without uncertainties, minimize the functional…

$$A[\mathcal{K}] \equiv \int_0^\infty d\omega\,(\bar{\omega}-\omega)^2\,\widehat{\delta}_\Delta(\bar{\omega},\omega)^2 = \int_0^\infty d\omega\,(\bar{\omega}-\omega)^2\left[\sum_\tau \mathcal{K}_\tau e^{-\omega\tau}\right]^2$$

with unit area constraint on δ

- Stabilized inverse

With uncertainties, instead minimize…

$$W_\lambda[\mathcal{K}] \equiv (1-\lambda)A[\mathcal{K}] + \lambda\frac{\mathcal{K}\cdot\mathrm{COV}\cdot\mathcal{K}}{\mathcal{N}}$$

wants oscillating K *wants well-behaved K*

λ parametrizes the tradeoff between ***final resolution*** and ***final* uncertainty**

Extended Backus Gilbert

- Important generalization from ***Martin** Hansen, Lupo, Tantalo*
- Addressed two limitations…

Shape and width of δ depends on data quality

No clear way to set λ

$$W_\lambda[\mathcal{K}] \equiv (1-\lambda)A[\mathcal{K}] + \lambda \frac{\mathcal{K}\cdot \mathrm{COV}\cdot \mathcal{K}}{\mathcal{N}}$$

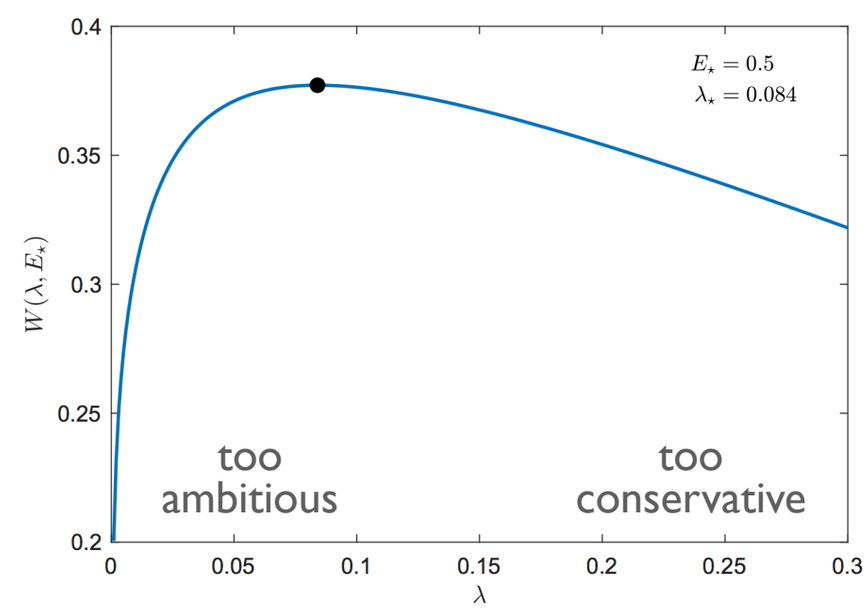
Change minimizing functional

$$A[\mathcal{K}] \equiv \int_0^\infty d\omega\, (\bar{\omega}-\omega)^2\, \widehat{\delta}_\Delta(\bar{\omega},\omega)^2$$

$$\widetilde{A}[\mathcal{K}] \equiv \int_0^\infty d\omega \left[\widehat{\delta}_\Delta^{\,\mathrm{target}}(\bar{\omega},\omega) - \widehat{\delta}_\Delta(\bar{\omega},\omega)\right]^2$$

Extremize *W w.r.t.* λ

$$W_\lambda[\mathcal{K}_{\min}(\lambda)]$$

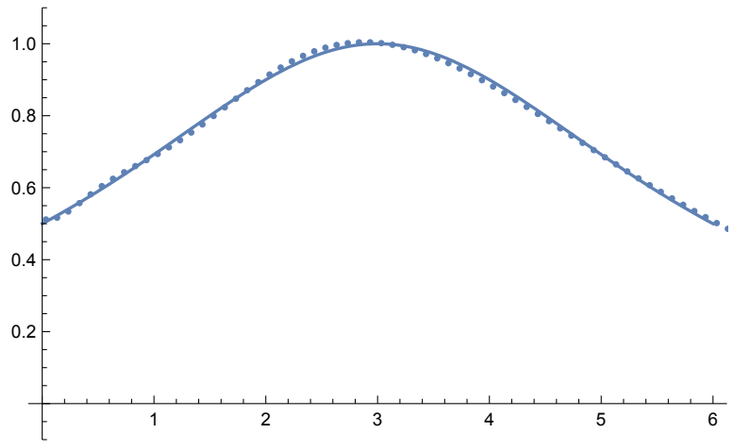


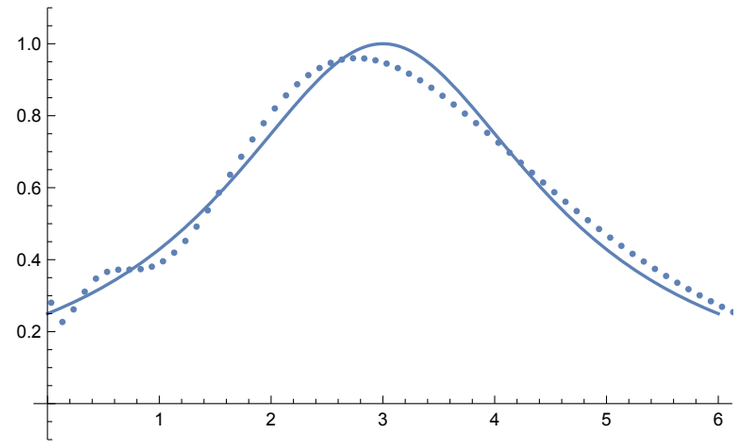
Hansen, Lupo, Tantalo (2019)

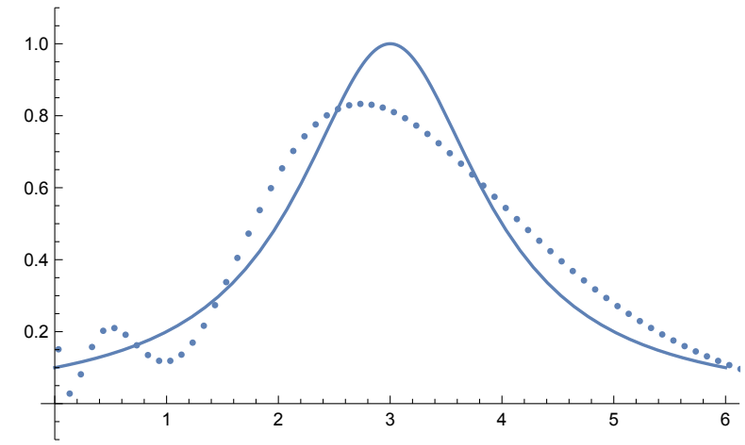
Conservation of evil

- Method will always fail if one tries to…

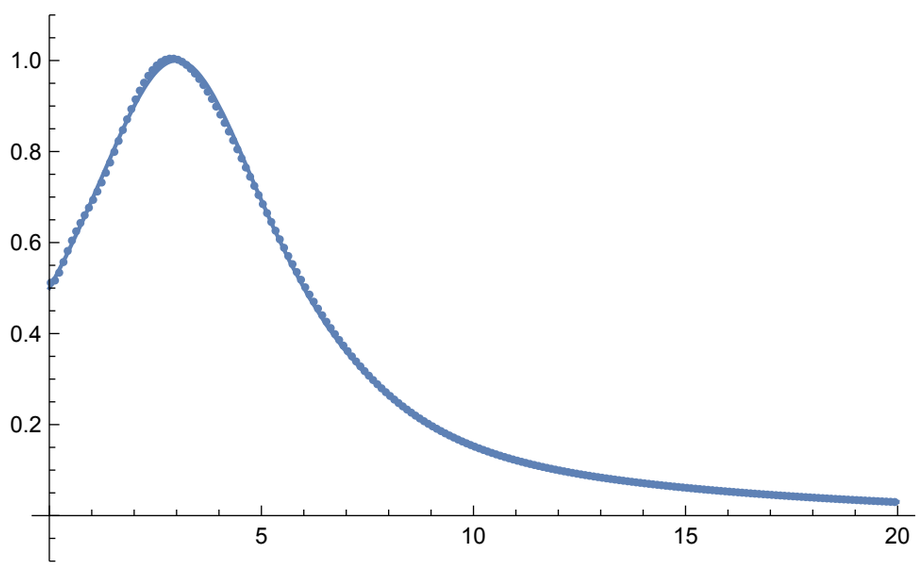
make the resolution function too narrow

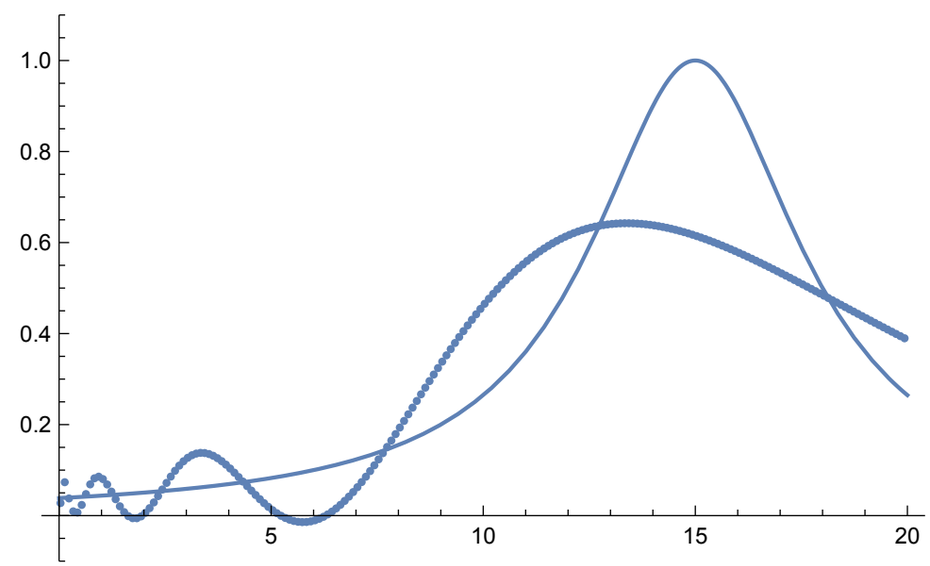






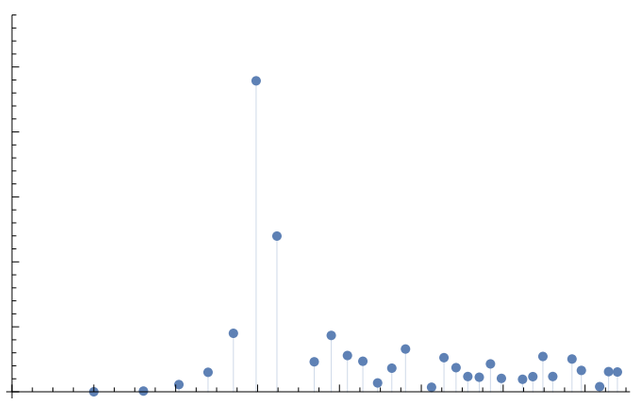
move the resolution too high in energy



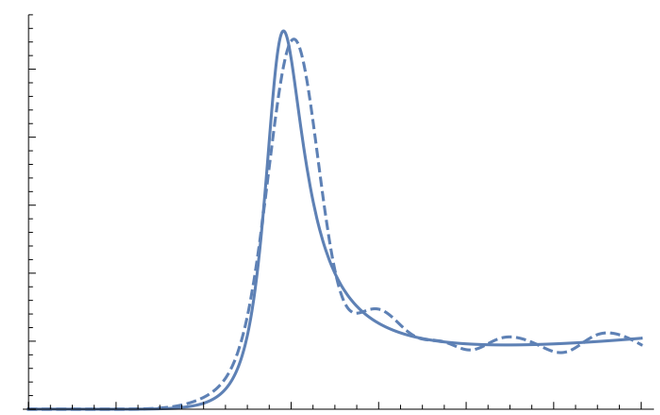


Intermediate summary

- Cannot solve the inverse problem, we can get $\widehat{\rho}_{L,\Delta}(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho_L(\omega)$
- Smearing is needed anyway to ***suppress volume effects***

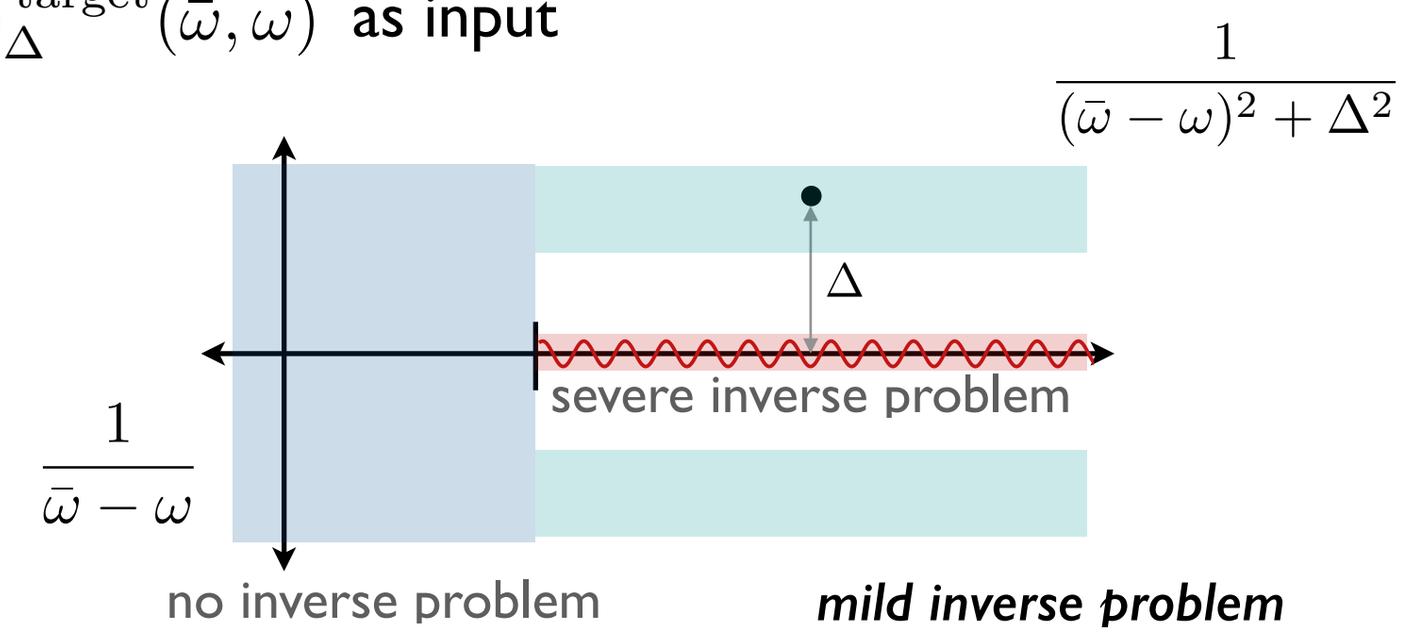


$$1/L \ll \Delta \ll \mu_{\text{physical}}$$



- Generalized Backus-Gilbert takes $\widehat{\delta}_\Delta^{\text{target}}(\bar{\omega},\omega)$ as input

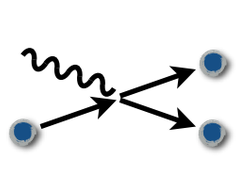
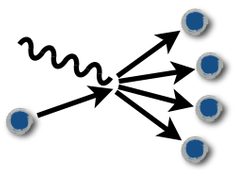
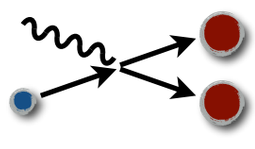
defines a continuum of inverse-problem severity

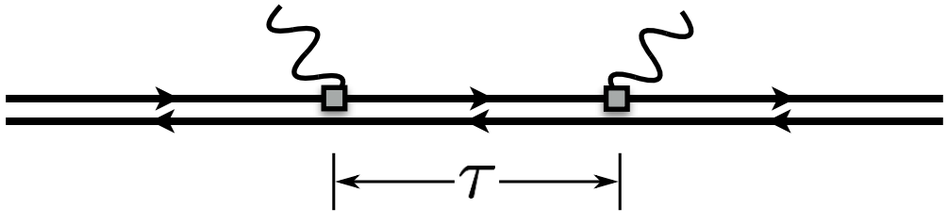


Total rate based applications

- Hadronic tensor, heavy flavor lifetimes

$$W_{\mu\nu}(p,q) \equiv \int d^4x \; e^{iqx} \langle \pi, p | \mathcal{J}_\mu^\dagger(x) \mathcal{J}_\nu(0) | \pi, p \rangle$$

$\propto \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \cdots$



$$= \int_0^\infty d\omega \, e^{-\omega\tau} \, W_{\mu\nu}(p,q)_{\omega = p^0 + q^0}$$

$$W_{\mu\nu} = \lim_{\Delta \to 0} \lim_{L \to \infty} \widehat{W}_{\mu\nu;\Delta,L}$$

- What about ***scattering*** and ***transition amplitudes?***

MTH, Meyer, Robaina (2017)

Amplitudes from spectral functions

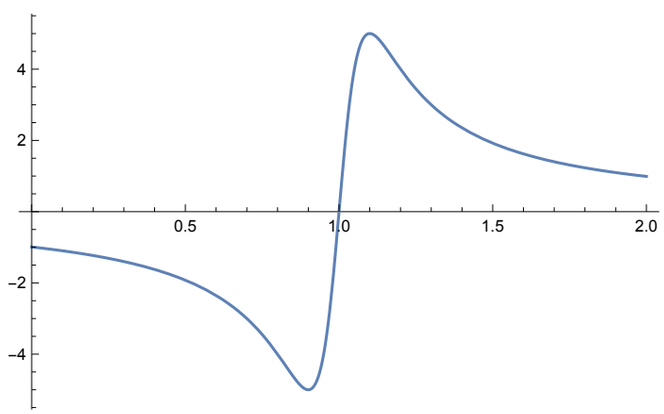
- First define the Euclidean correlator

$$G_L(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi(\tau_3, \boldsymbol{p}_3) \pi(0) | \pi_{\boldsymbol{p}_1} \rangle_L = \int_0^\infty dE_3 \, \rho(E_3) e^{-E_3 \tau_3}$$

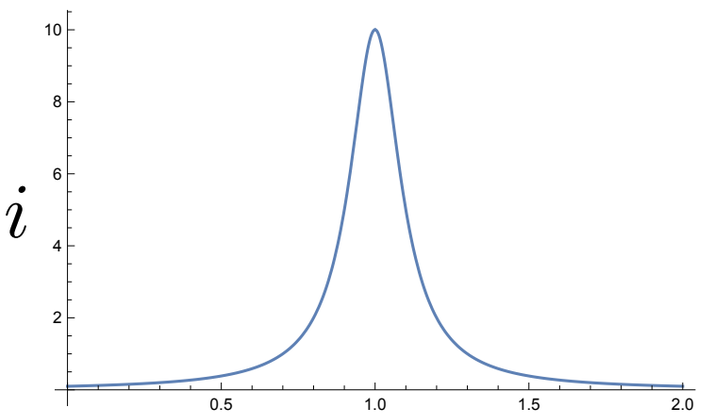
and the smeared spectral function

$$\widehat{\rho}^{L,\epsilon}_{\boldsymbol{p}_4 \boldsymbol{p}_1}(q_3) = \int_0^\infty dE_3 \, \frac{1}{q_3^0 - E_3 + i\epsilon} \, \rho(E_3) = \int_0^\infty dE_3 \, \widehat{\delta}_\epsilon(q_3^0, E_2) \, \rho(E_3)$$

$$\frac{1}{q_3^0 - E_3 + i\epsilon} =$$



$$-i$$



Bulava, MTH (2019)

Amplitudes from spectral functions

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- Next project *on shell* at finite ϵ

$$\mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1) \equiv \frac{2E(\boldsymbol{p}_3)}{Z^{1/2}(\boldsymbol{p}_3)} \frac{2E(\boldsymbol{p}_2)}{Z^{1/2}(\boldsymbol{p}_2)} \epsilon^2 \, \widehat{\rho}^{L,\epsilon}_{\boldsymbol{p}_4 \boldsymbol{p}_1}(E(\boldsymbol{p}_3), \boldsymbol{p}_3)$$

- Finally project out the ***scattering amplitude***

$$\mathcal{M}_{\mathrm{c}}(p_4 p_3 | p_2 p_1) = \lim_{\epsilon \to 0^+} \lim_{L \to \infty} \mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1)$$

Bulava, MTH (2019)

Some comments

- Derivation based in modified **LSZ** + ***signature-independence*** of ρ(E)
- Holds when LSZ holds

$$\langle m, \mathrm{out} | n, \mathrm{in} \rangle \qquad \langle m, \mathrm{out} | \mathcal{J}(0) | n, \mathrm{in} \rangle$$

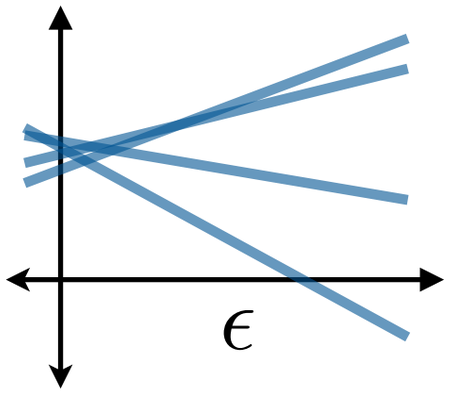
- Very challenging… but systematic

  for some (unknown) volume + correlator quality, we can get $D \to \pi\pi, K\overline{K}$

- Some nice features…

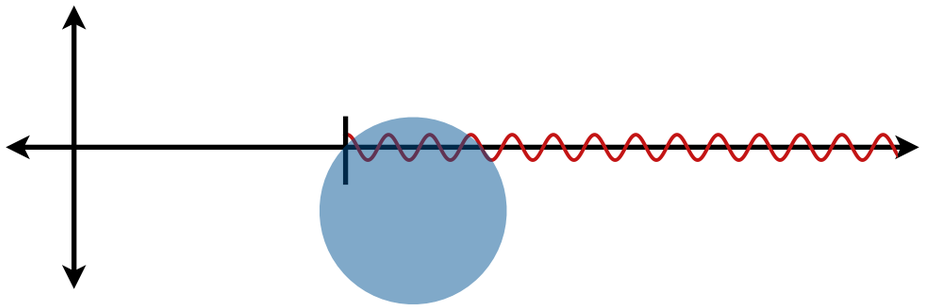
  GEVP-like operator freedom

$$G_L^{[ab]}(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi^a(\tau_3, \boldsymbol{p}_3) \pi^b(0) | \pi_{\boldsymbol{p}_1} \rangle_L \longrightarrow \mathcal{M}^{[ab],\epsilon,L}$$



Finite range of analyticity in ϵ

Uses overlaps



Bulava, MTH (2019)

Perturbative study…

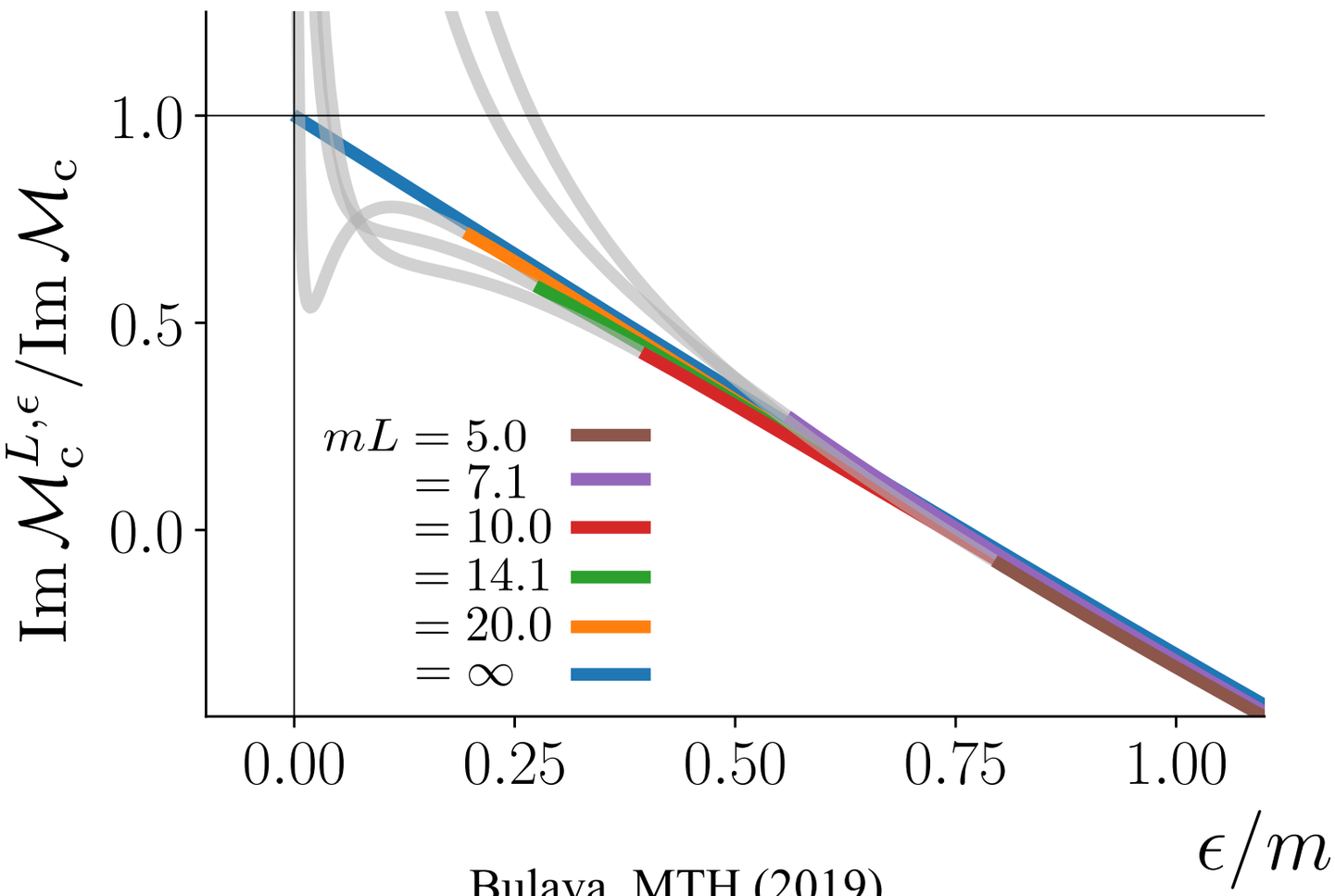
Calculate in PT

$$G_L(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi(\tau_3, \boldsymbol{p}_3) \pi(0) | \pi_{\boldsymbol{p}_1} \rangle_L$$

Convert to this

$$\mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1)$$

$$\mathrm{Im}\, \mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1) = \frac{\lambda^2}{2} \frac{1}{L^3} \sum_{\boldsymbol{k}'}^{\Lambda} \frac{1}{(2E(\boldsymbol{k}'))^2} \mathrm{Im} \left\{ \frac{1}{(E_{\mathrm{cm}} - 2E(\boldsymbol{k}') + i\epsilon)} \left[1 - \frac{\epsilon^2}{4E(\boldsymbol{k}')^2} - \frac{\epsilon(\epsilon + 2iE(\boldsymbol{k}'))}{E_{\mathrm{cm}} E(\boldsymbol{k}')} \right] \right\}$$



Bulava, MTH (2019)

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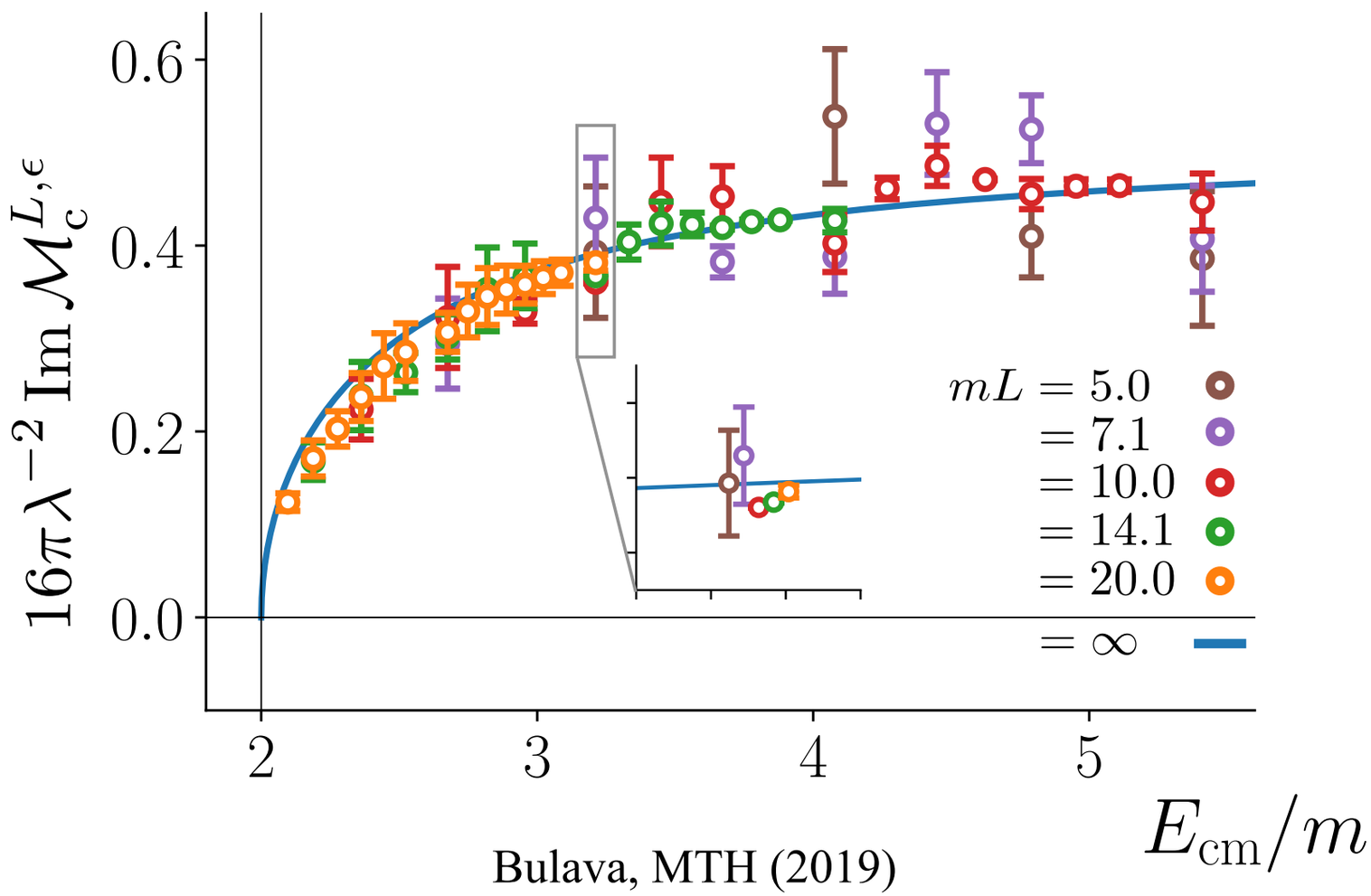


Bulava, MTH (2019)

Connection to Maiani-Testa

- Maiani and Testa considered correlators of the form

$$G_{[\boldsymbol{p}]}(\tau) = \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(\tau) J(0) | 0 \rangle$$

- And showed two key points

$$G_{[\mathbf{0}]}(\tau) = \frac{\sqrt{Z_\pi}}{2M_\pi} e^{-M_\pi \tau} f(4M_\pi^2) \left[1 - a_{\pi\pi} \sqrt{\frac{m_\pi}{\pi\tau}} + O(\tau^{-3/2}) \right]$$

$G_{[\boldsymbol{p}\neq\mathbf{0}]}(\tau)$ is plagued by un-physical (operator dependent) contributions

- Recent work with M. Bruno connects this story to spectral function amplitudes

Variations on the Maiani-Testa approach and the inverse problem

M. Bruno[a] and M. T. Hansen[b]

arXiv: 2012.11488

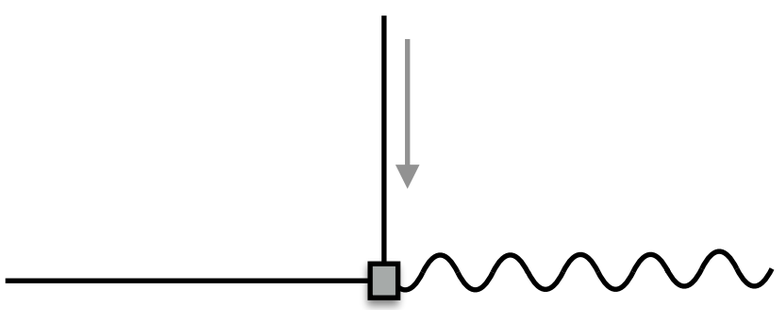
G is a smeared spectral function

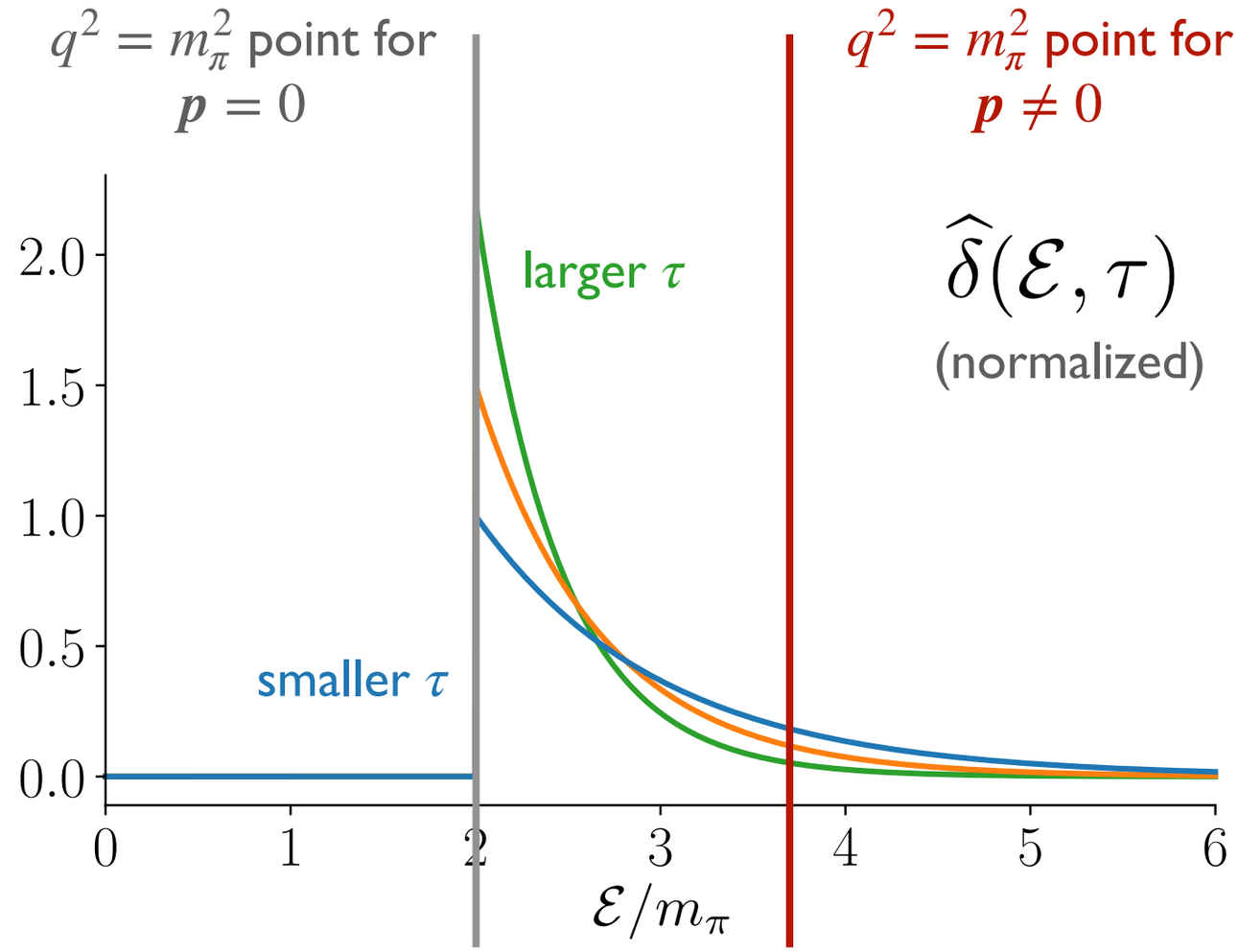
- Correlator can be viewed as a smeared spectral function

$$G_{[\boldsymbol{p}]}(\tau) = \int_0^\infty d\mathcal{E}\, \rho(\mathcal{E})\, \widehat{\delta}(\mathcal{E}, \tau) \qquad \widehat{\delta}(\mathcal{E}, \tau) = \theta(\mathcal{E} - 2m_\pi) \exp[-\mathcal{E}\tau]$$

$$\rho(\mathcal{E})$$

$$q^2 = (\mathcal{E} - \omega_{\boldsymbol{p}})^2 - \boldsymbol{p}^2$$

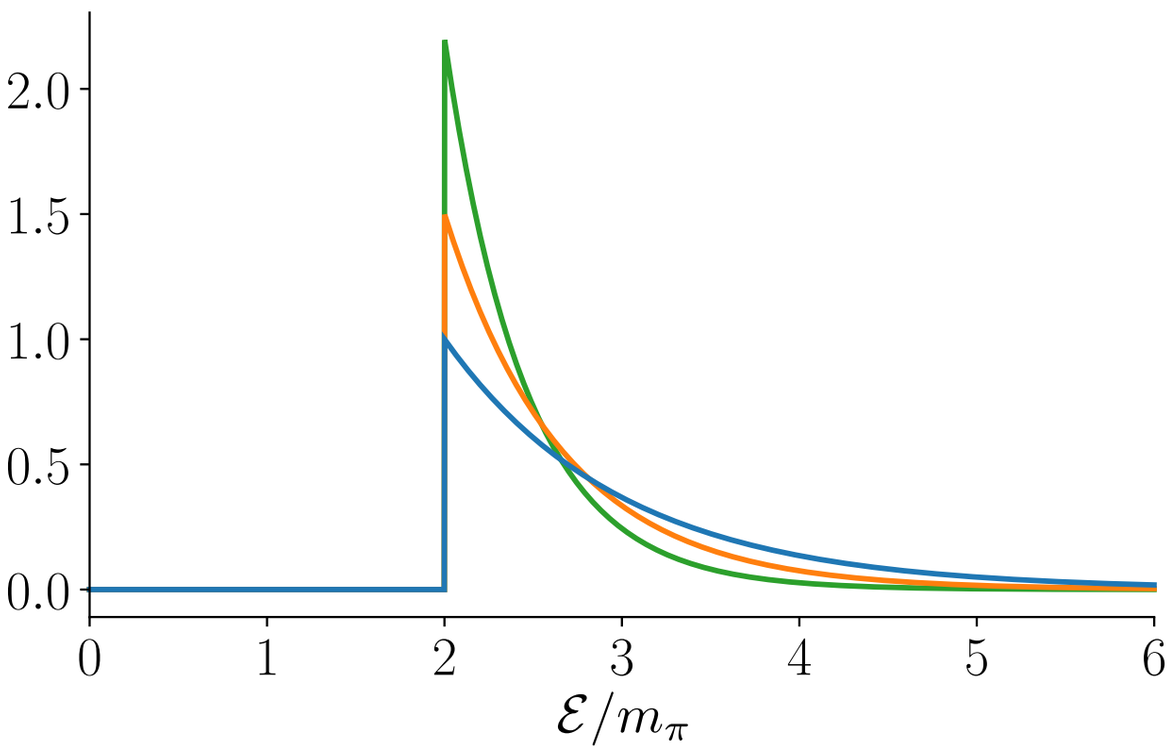




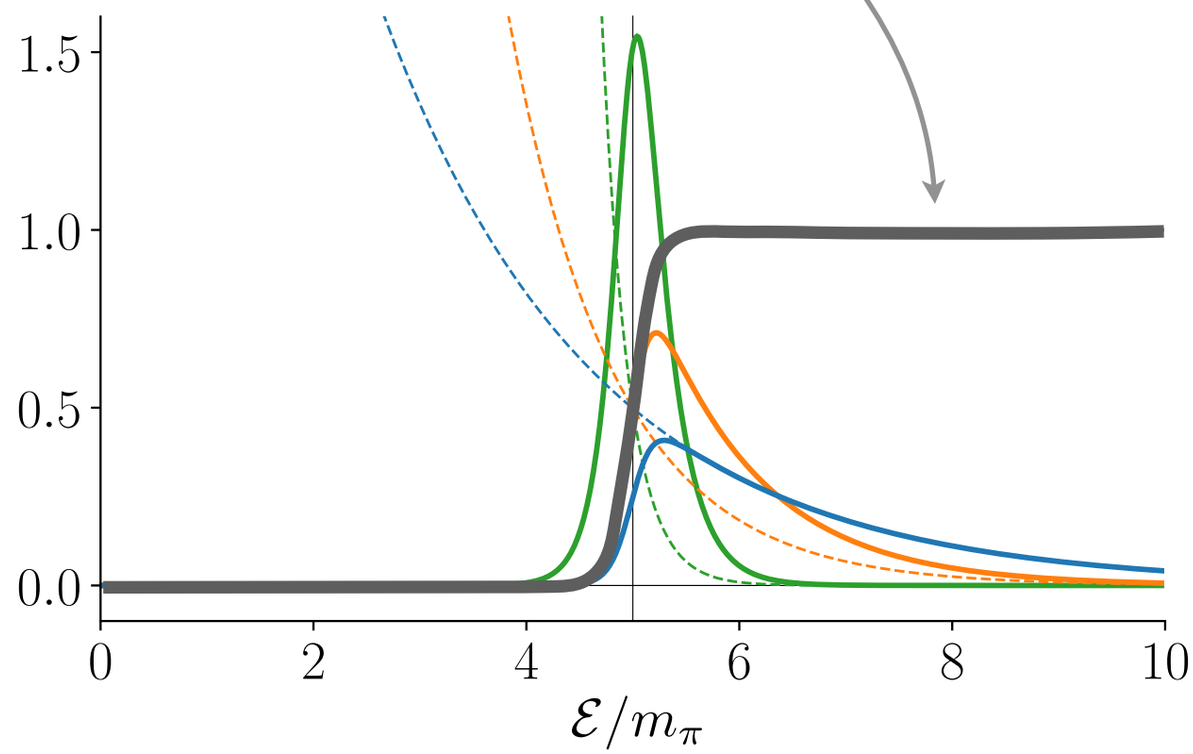
Bruno, MTH (2020)

Shift the peak!

$$\widehat{\delta}(\mathcal{E},\tau) = \theta(\mathcal{E} - 2m_\pi)\exp[-\mathcal{E}\tau]$$



$$\widehat{\delta}^{\Theta}(\mathcal{E},\tau) = \Theta(\mathcal{E} - 2m_\pi, \Delta)\exp[-\mathcal{E}\tau]$$



Bruno, MTH (2020)

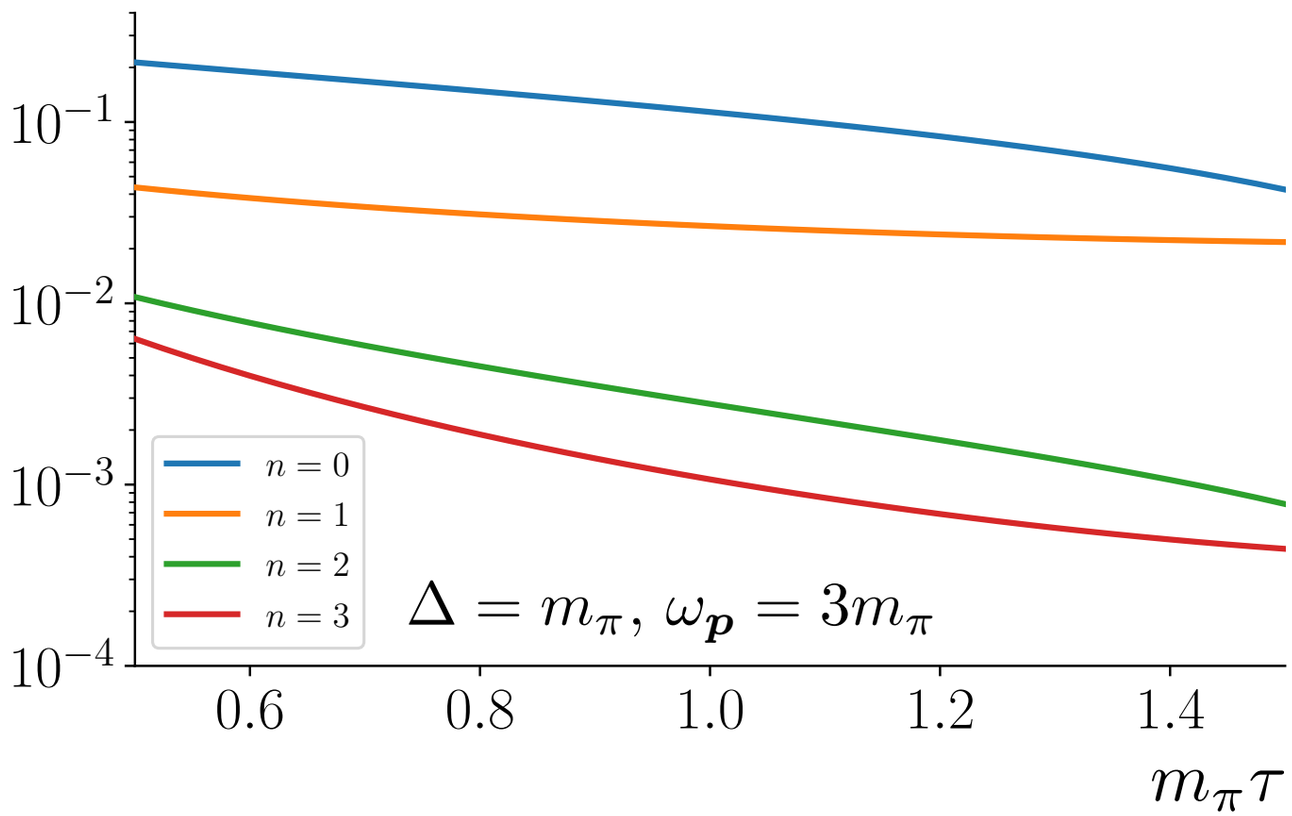
Reaching above threshold

- So the Maiani and Testa correlator becomes

$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(\tau)\, \Theta(\hat{H} - 2\omega_{\boldsymbol{p}}, \Delta)\, J(0) | 0 \rangle$$

- Now separating the fields gives something useful above threshold

$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \frac{\sqrt{Z_\pi}}{2\omega_{\boldsymbol{p}}} e^{-\omega_{\boldsymbol{p}}\tau} \left[\Theta(0,\Delta)\, \mathrm{Re} f(4\omega_{\boldsymbol{p}}^2) - 2\mathcal{J}^{(0)}(\tau,\Delta)\, \mathrm{Im} f(4\omega_{\boldsymbol{p}}^2) + \cdots \right]$$



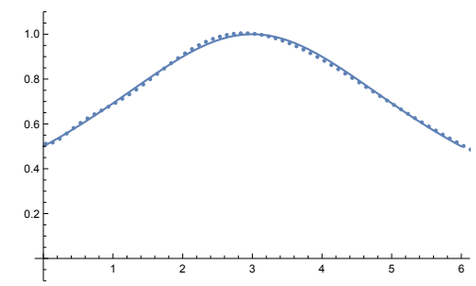
Hierarchy of $\mathcal{J}^{(n)}$ provides a useful fit function

known functional forms = distinction from the work with Bulava

Constructing the Θ correlator

$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(\tau)\, \Theta(\hat{H} - 2\omega_{\boldsymbol{p}}, \Delta)\, J(0) | 0 \rangle$$

- Two main methods:
  - Backus-Gilbert and HLT method
  - GEVP



$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \sum_n \Theta(E_n - 2\omega_{\boldsymbol{p}}, \Delta) e^{-(E_n - \omega_{\boldsymbol{p}})\tau} \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(0) | n \rangle \langle n | J(0) | 0 \rangle$$

combine finite-volume energies and matrix elements

- Volume effects?… Suppressed as $e^{-\Delta L}$, but more investigation is needed

Five-day outline

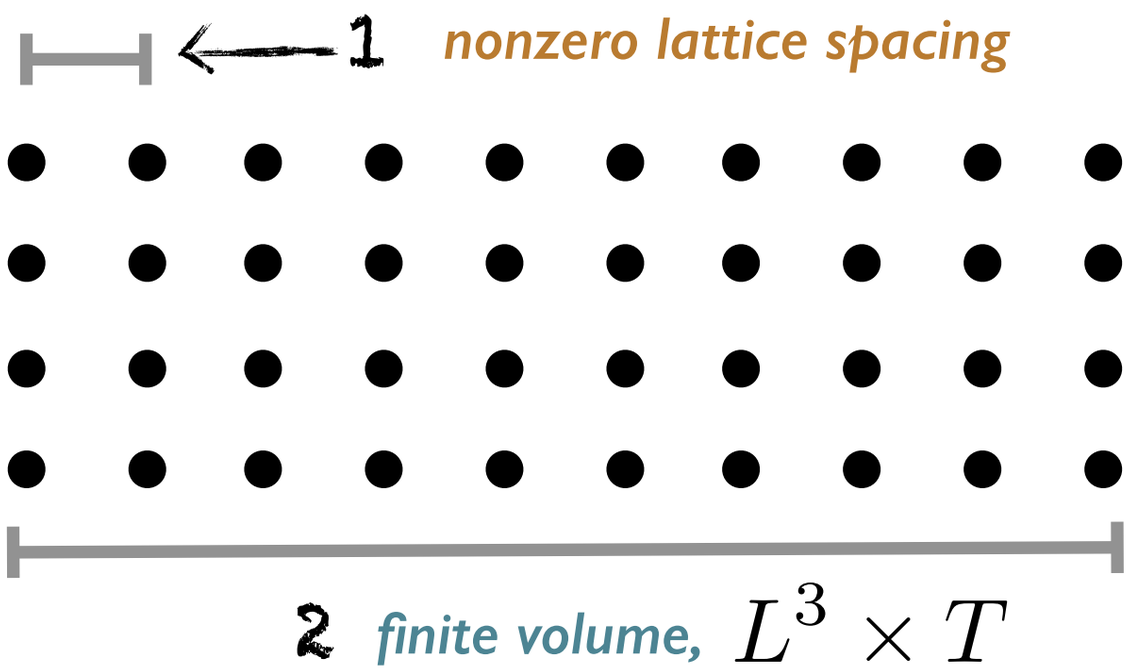
- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} volume effects
  - ☑ Mass, matrix element in $\lambda\phi^4, g\phi^3$
  - ☑ Pion mass
  - ☑ LO-HVP for $(g-2)_\mu$
  - ☑ Bethe-Salpeter kernel
- ☑ $2 \to 2$ finite-volume (f.v.) formalism
  - ☑ Scattering basics
  - ☑ Derivation
  - ☑ Generalizations
  - ☑ Applications
- ☑ $(1+)\mathscr{J} \to 2$ f.v. formalism
  - ☑ Derivation
  - ☑ Expansions
  - ☑ Applications
- ☑ $2 + \mathscr{J} \to 2$ f.v. formalism
  - ☑ Derivation
  - ☑ Testing the result
  - ☑ Numerical explorations
- ☑ Non-local matrix elements
  - ☑ Derivation
  - ☑ Applications
- ☑ $3 \to 3$ f.v. formalism
  - ☑ New complications
  - ☑ Derivation ($E_n(L)$ to $\mathscr{K}_{\mathrm{df},3}$)
  - ☑ Integral equations ($\mathscr{K}_{\mathrm{df},3}$ to $\mathscr{M}_3$)
  - ☑ Testing the result
  - ☑ Numerical explorations/calculations
- ☑ Finite-volume QED + QCD
  - ☑ Different formalisms
  - ☑ Basic examples
- ☑ Large-volume ideas
  - ☑ Spectral functions
  - ☑ Maiani-Testa

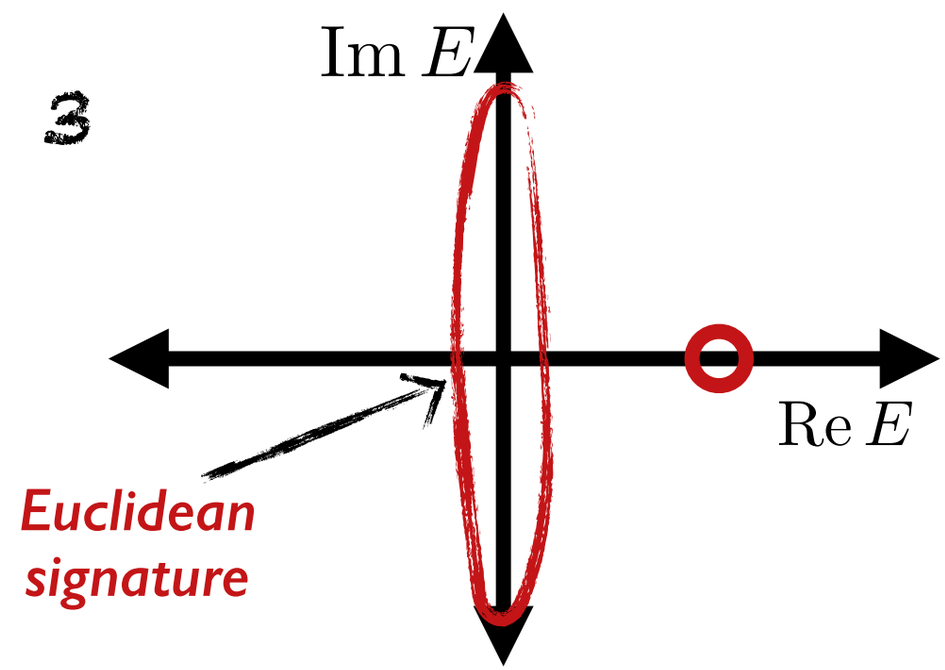
Recap!

Three essential modifications

$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

To proceed we have to make *three modifications*

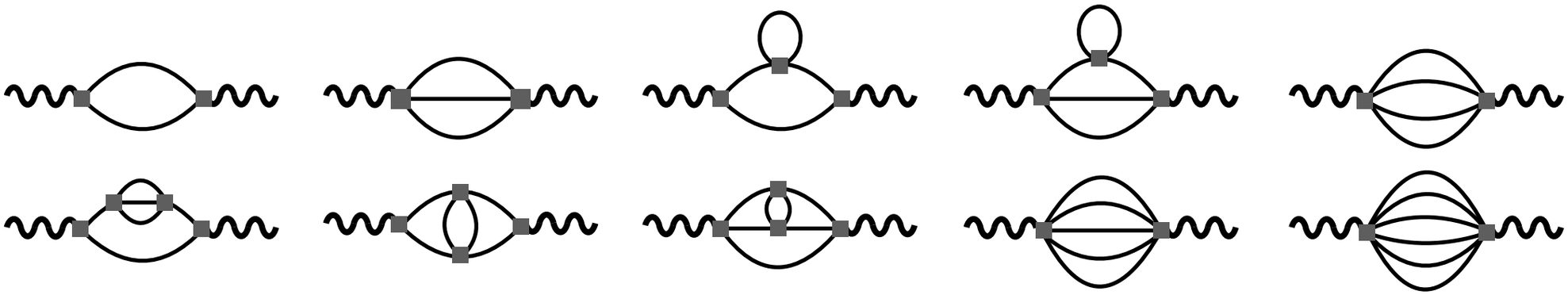




Also… $M_{\pi,\mathrm{lattice}} > M_{\pi,\mathrm{our\ universe}}$

(but physical masses → increasingly common)

Exponentially suppressed volume effects

$a_\mu(L) =$  $+ \cdots$

$$\Delta a_\mu(L) = -\frac{\alpha^2}{m_\mu^2} \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \int \frac{dp_3}{2\pi} \frac{e^{-|\boldsymbol{n}|L\sqrt{M_\pi^2+p_3^2}}}{2\pi L|\boldsymbol{n}|} \int_0^\infty dx_0\, \widehat{\mathcal{K}}(m_\mu x_0) \int \frac{dk_3}{2\pi} \cos(x_0 k_3)\, \mathrm{Re}\, T(-k_3^2, -k_3 p_3)$$

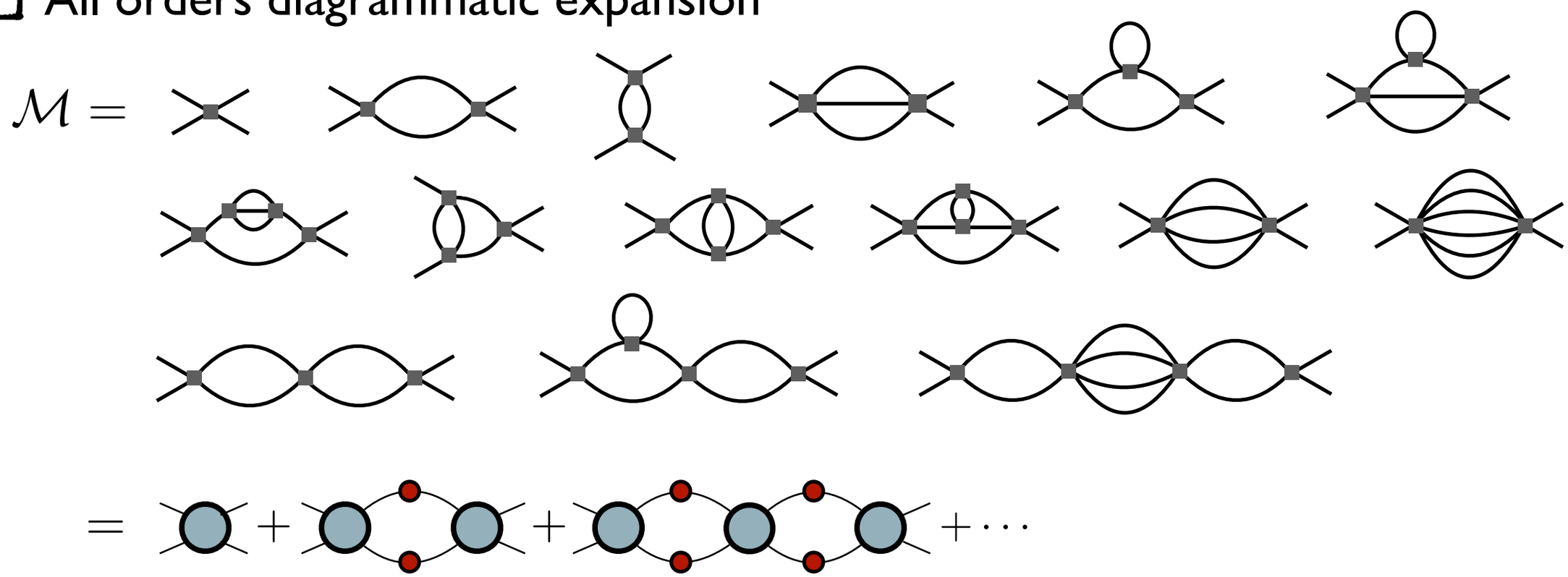
volume dependence — kernel — Compton amplitude

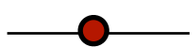
$$T(k^2, k\cdot p) \equiv i \lim_{\boldsymbol{p}'\to\boldsymbol{p}} \sum_{q=0,\pm1} \int d^4x\, e^{ikx}\, \langle \boldsymbol{p}', q|\mathrm{T}\mathcal{J}_\rho(x)\mathcal{J}^\rho(0)|\boldsymbol{p}, q\rangle$$

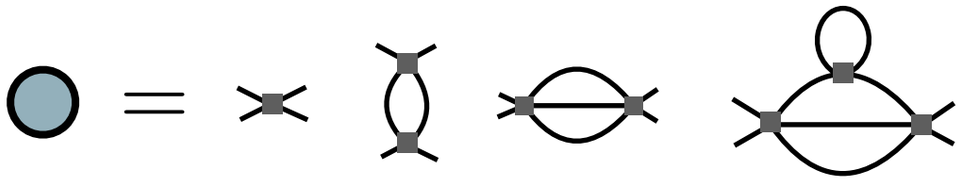
reference to EFT has disappeared — spacelike photons only

$2 \to 2$: Bethe Salpeter equation

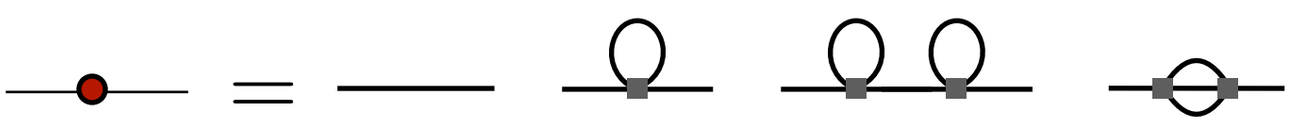
- All orders diagrammatic expansion



- Construct  and  such that this is true



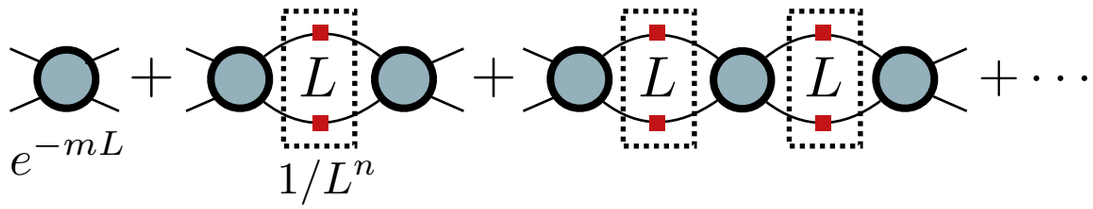
Bethe Salpeter kernel
(2PI in the s-channel)



Fully dressed propagator

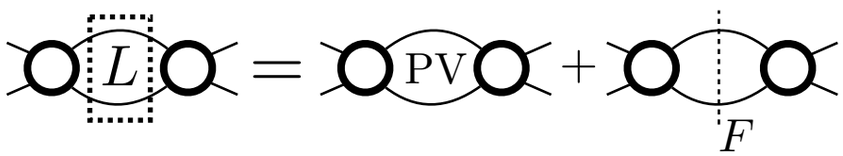
$2 \to 2$: Derivation

- Consider the finite-volume correlator:

$\mathcal{M}_L(P) =$ 

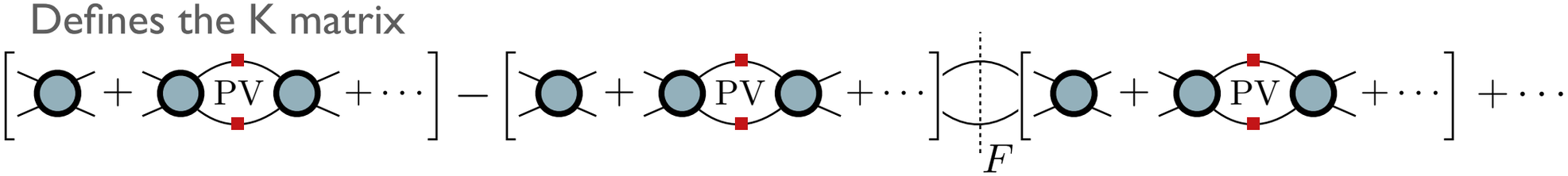
$= \sum_{\mathbf{k}}$

For two-particle energies $(2m)^2 < s < (4m)^2$, what is the *L* dependence?

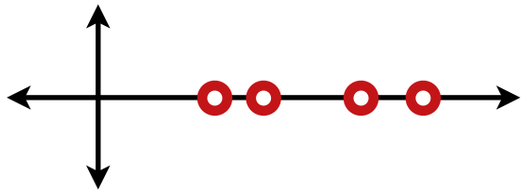


Cut projects loop to ***on-shell energies***

Defines the K matrix



$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$

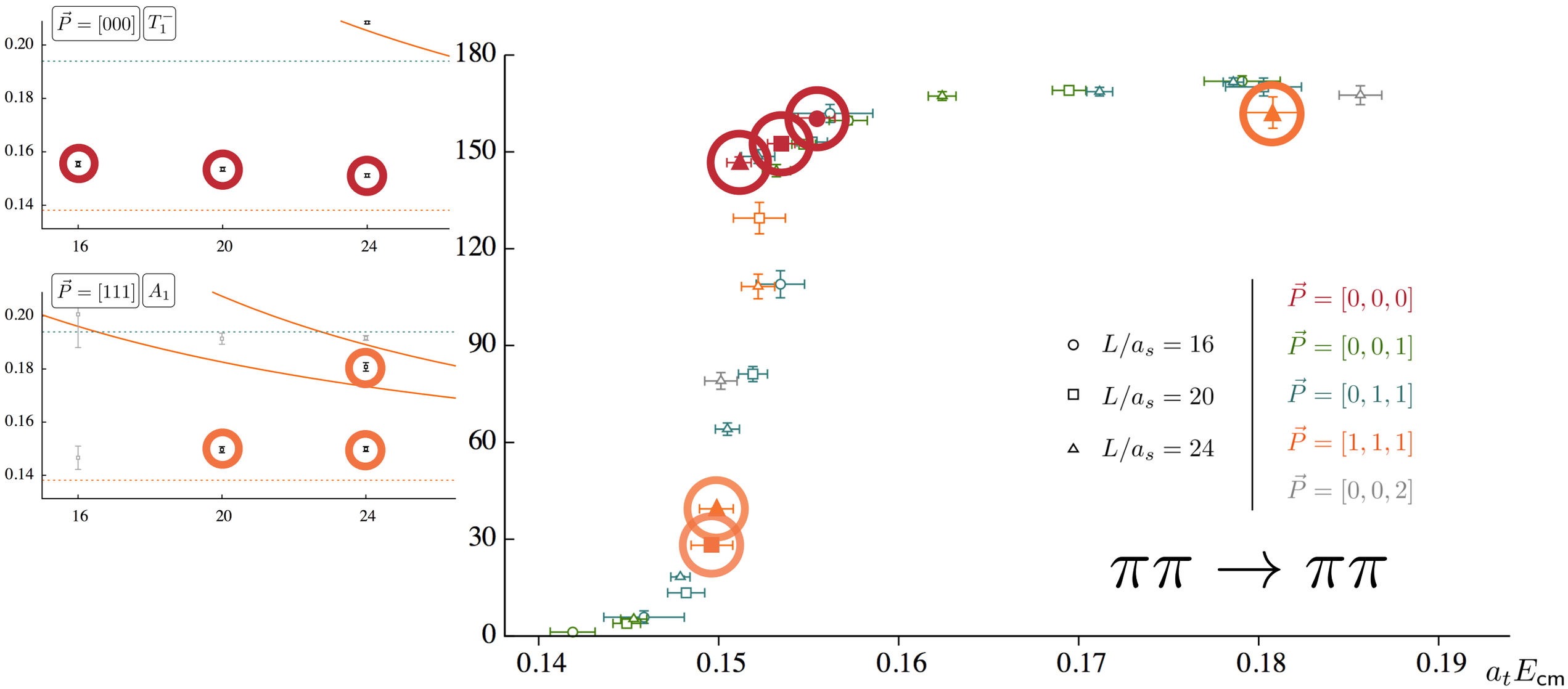


- Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (*coupled channels*, 2012) •

2 → 2: Applications

- Single-channel case *(pions in a p-wave)*

$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$



- Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

$1 + \mathscr{J} \to 2$

- Finite-volume matrix element

$$C_L(P) = \int_L d^4x\, e^{-iPx}\, \langle \pi | \mathrm{T}\{\mathcal{J}(x)\mathcal{J}(0)\} | \pi \rangle_L$$

Insert a complete set finite-volume of states

$$\xrightarrow[E \to E_n]{} -\frac{L^3 \langle \pi | \mathcal{J}(0) | n, L \rangle \langle n, L | \mathcal{J}(0) | \pi \rangle}{E - E_n(L)}$$

- Infinite-volume matrix element

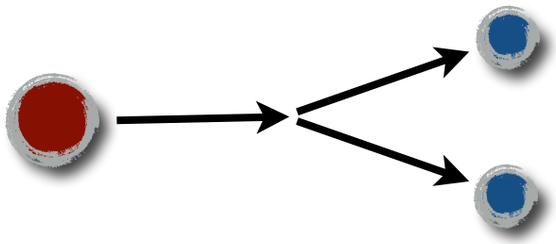
$$C_L(P) = C_\infty^{i\epsilon}(s) - A_{\text{in}}^{i\epsilon}(s) \frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} A_{\text{out}}^{i\epsilon}(s)$$

$$\xrightarrow[E \to E_n]{} -\frac{\langle \pi | \mathcal{J}(0) | \pi\pi, \text{in} \rangle\, \mathcal{R}(E, \boldsymbol{P}, L)\, \langle \pi\pi, \text{out} | \mathcal{J}(0) | \pi \rangle}{E - E_n(L)}$$

Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016)

$1 + \mathcal{J} \to 2$

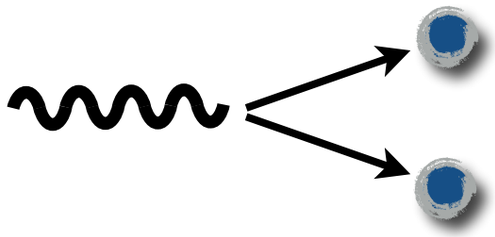
Kaon decay

$\langle \pi\pi, \mathrm{out} | \mathcal{H} | K \rangle \equiv$ 

Implementation by RBC/UKQCD collaboration

Lellouch, Lüscher (2001) • Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005)

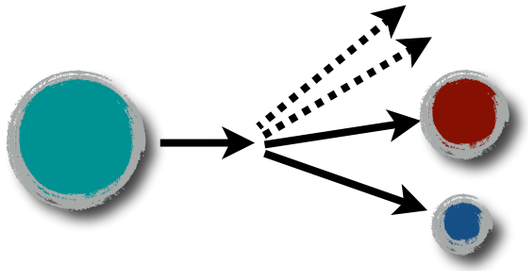
Time-like form factors

$\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | 0 \rangle \equiv$ 

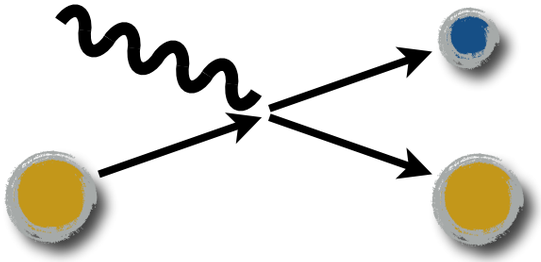
Relevant for muon HVP contribution to muon g-2

Meyer (2011)

Resonance transition amplitudes

$\langle K\pi, \mathrm{out} | \mathcal{J}_{\alpha\beta} | B \rangle \equiv$ 

Particles with spin

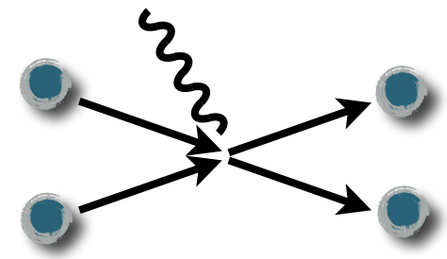
$\langle N\pi, \mathrm{out} | \mathcal{J}_\mu | N \rangle \equiv$ 

Agadjanov *et al*. (2014) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016)

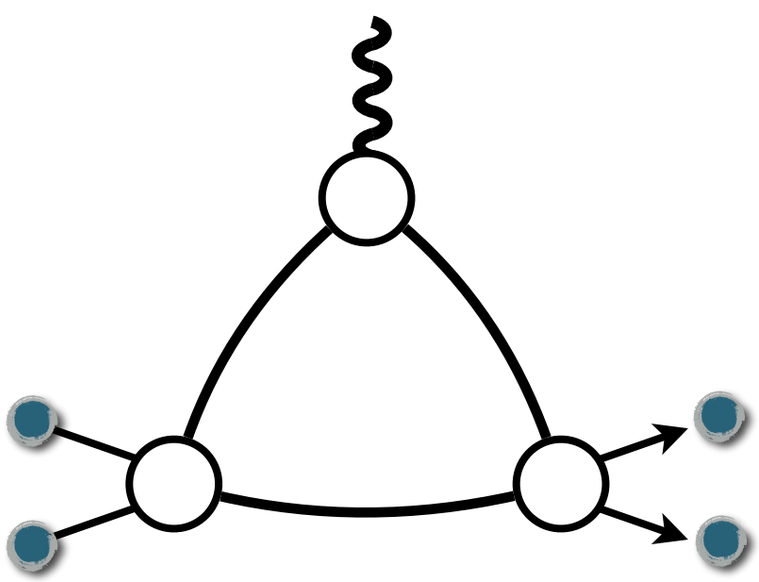
$2 + \mathscr{J} \to 2$

- Fully developed formalism for multi-hadron form factors

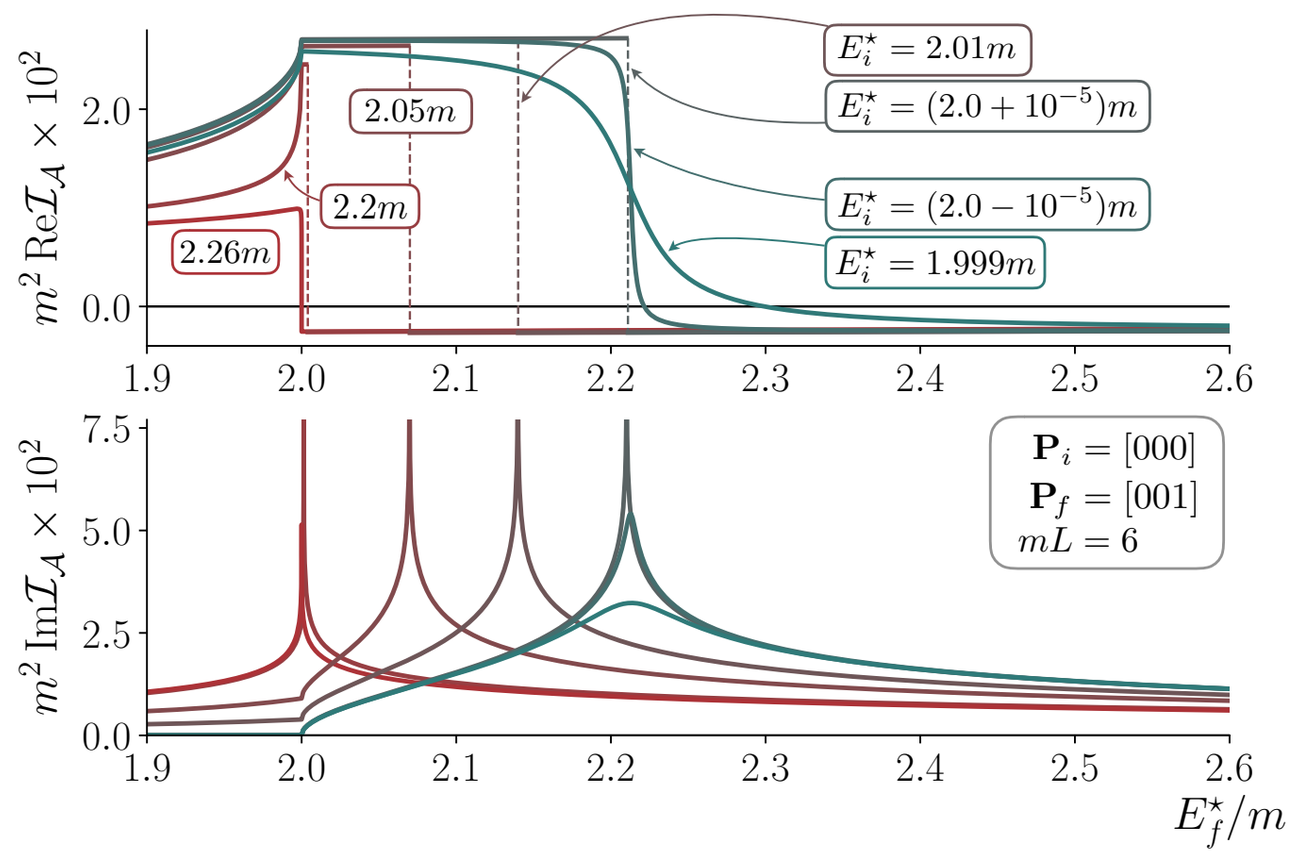
$$\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi\pi, \mathrm{in} \rangle \equiv$$



- Continuation to the pole → ***resonance form factors***
- Must carefully treat ***triangle*** singularities

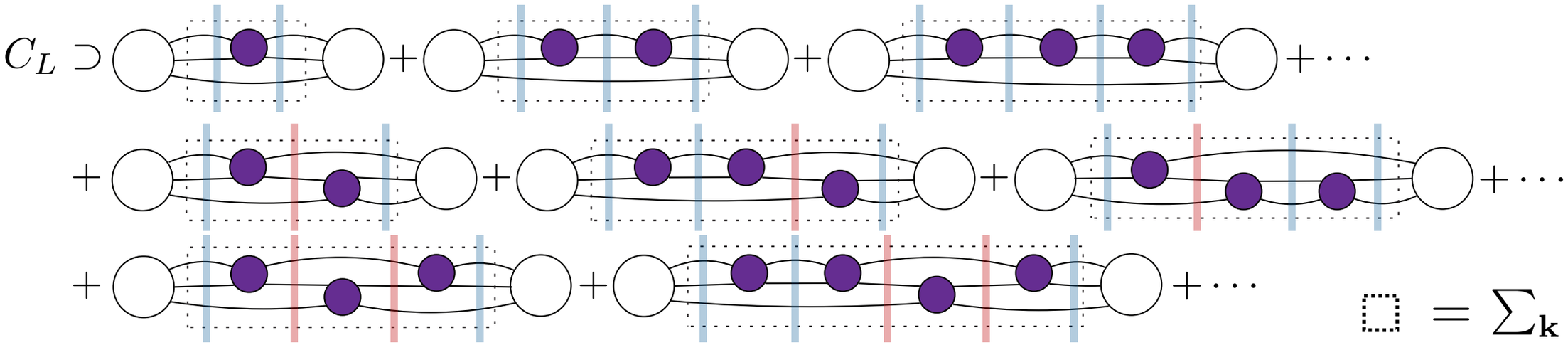


$\sim$



Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

Two types of cuts



$$\mathbf{A}_3' \mathbf{F} \mathbf{K}_2 \mathbf{F} \mathbf{A}_3 + \mathbf{A}_3' \mathbf{F} [\mathbf{K}_2 \mathbf{F}]^2 \mathbf{A}_3 + \mathbf{A}_3' \mathbf{F} [\mathbf{K}_2 \mathbf{F}]^3 \mathbf{A}_3 + \cdots = \mathbf{A}_3' \mathbf{F} \frac{1}{1 - \mathbf{K}_2 \mathbf{F}} \mathbf{K}_2 \mathbf{F} \mathbf{A}_3$$

$$\mathbf{A}_3' \mathbf{F} \mathbf{K}_2 \mathbf{G} \mathbf{K}_2 \mathbf{F} \mathbf{A}_3 + \cdots$$

$$C_L - C_\infty \supset \mathbf{A}_3' \mathbf{F}_{33} \mathbf{A}_3$$

$$\mathbf{F}_{33} \equiv \frac{1}{3} \mathbf{F} + \mathbf{F} \mathbf{K}_2 \frac{1}{1 - (\mathbf{F} + \mathbf{G}) \mathbf{K}_2} \mathbf{F}$$

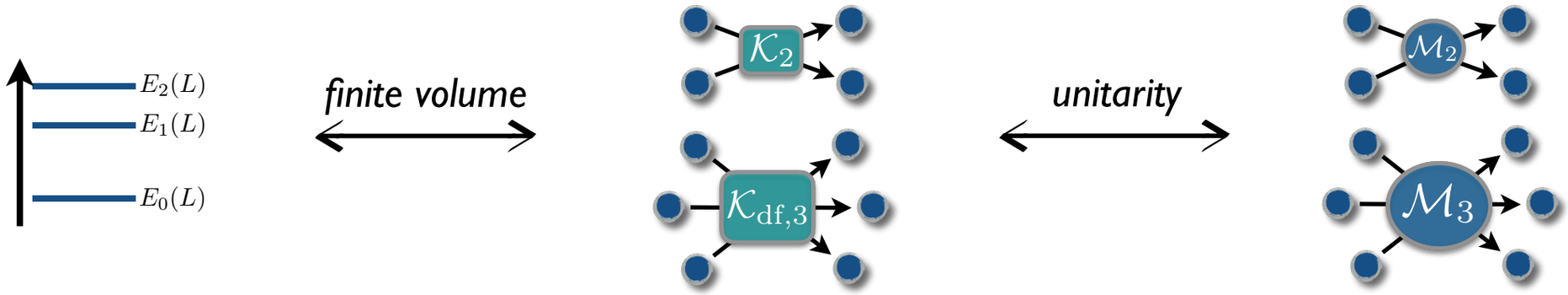
single **F** requires special treatment (1/3)

have not yet considered entire diagram contributions

missing contributions from *off-shellness*

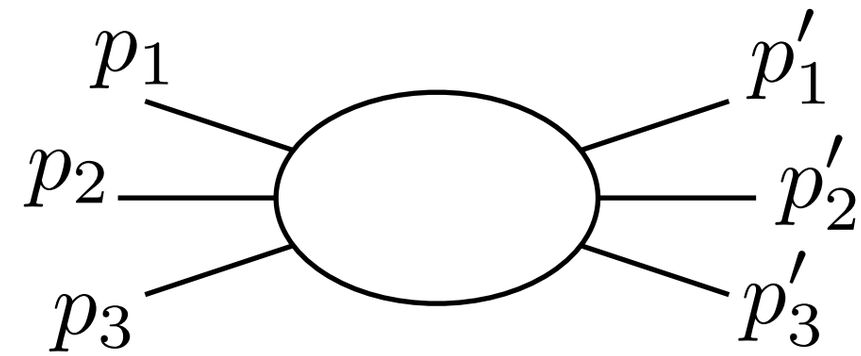
missing smooth terms (short-distance parts)

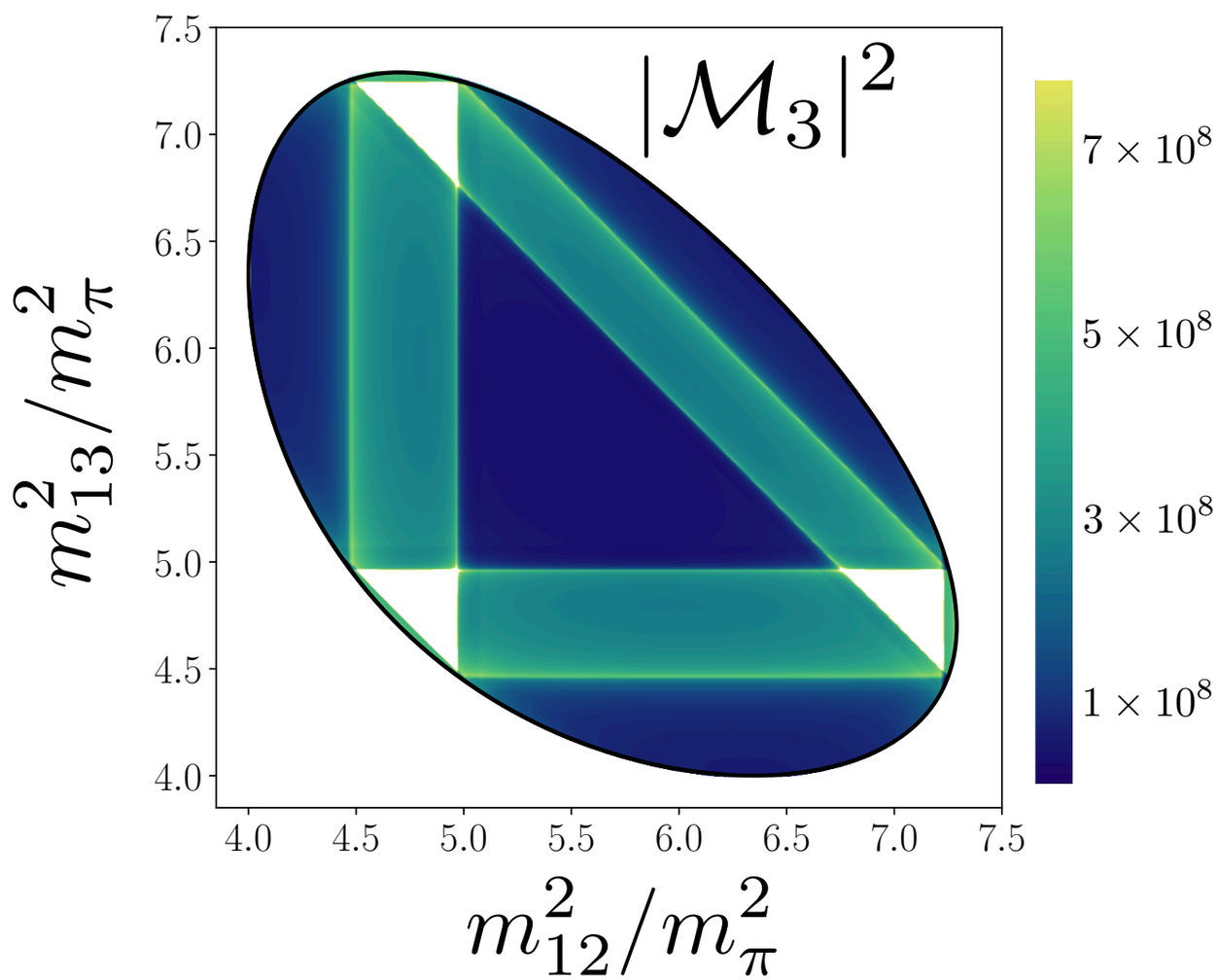
Status…

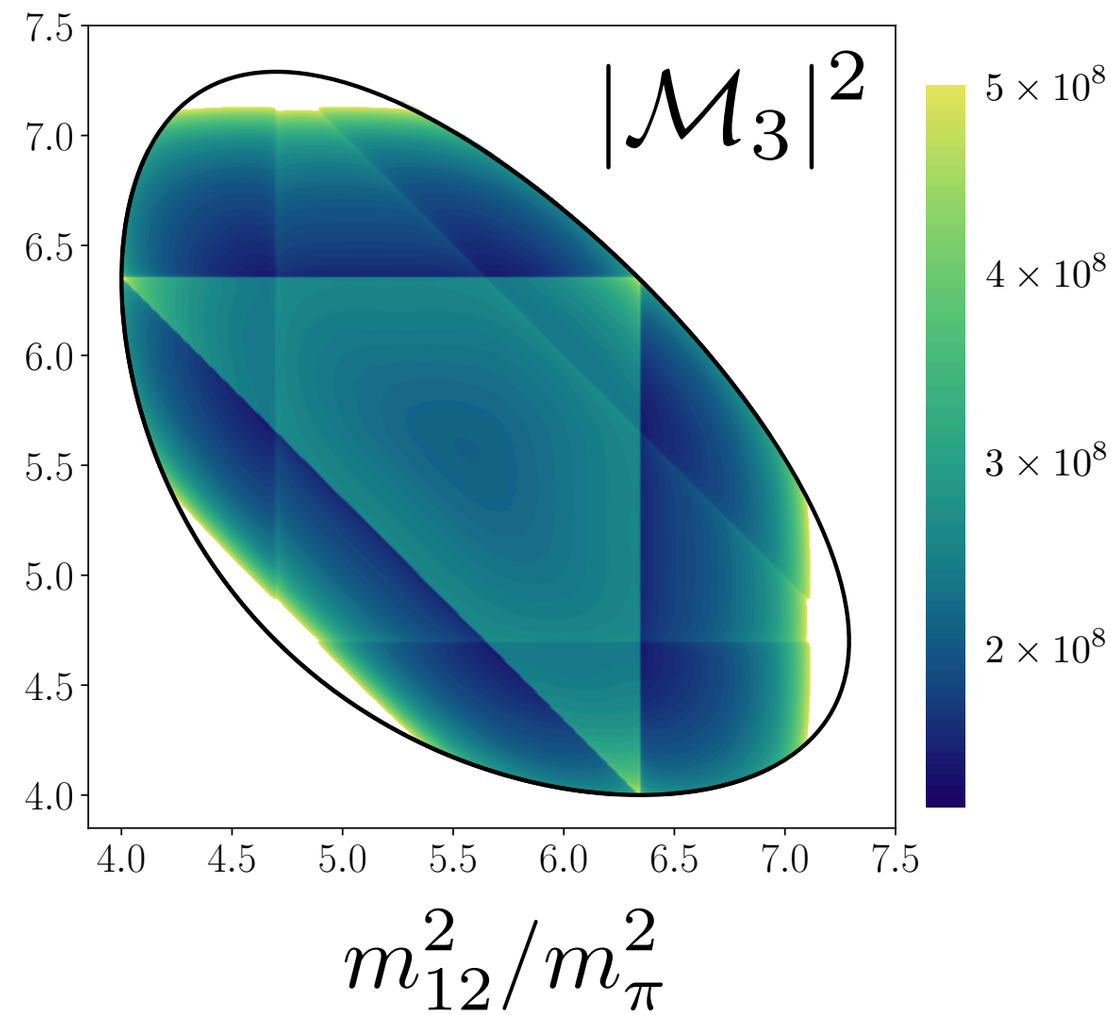


- ☑ Identical spin-zero, no 2-to-3, no K2 poles • MTH, Sharpe (2014, 2015) •
- ☑ as above… but including 2-to-3 • Briceño, MTH, Sharpe (2017) •
- ☑ as above… but including K2 poles • Briceño, MTH, Sharpe (2018) •
- ☑ Non-identical, non-degenerate spin-zero $\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$
  • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020, 2021)
- ☐ Multiple three-particle channels… Spin!

$$\mathcal{M}_3 = \sum_{i,j\in\{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p_i', p_j)$$





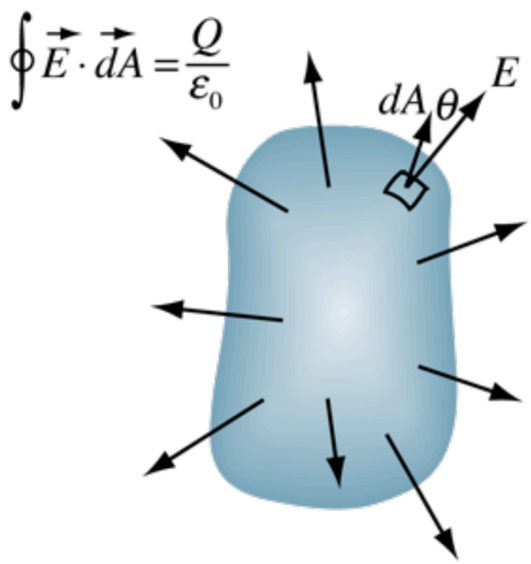


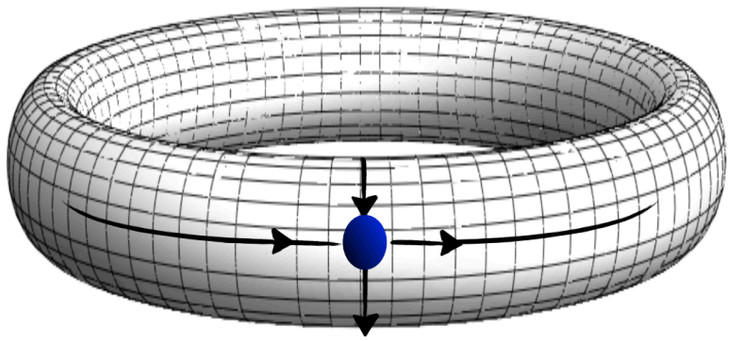
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

QED in a box

Gauss's law prohibits a naive implementation of charged objects in a periodic volume

$$Q = \int d^3\boldsymbol{x}\, j_0(t,\boldsymbol{x}) = \int d^3\boldsymbol{x}\, \nabla\cdot\boldsymbol{E}(t,\boldsymbol{x}) = \int d\boldsymbol{S}\cdot\boldsymbol{E}(t,\boldsymbol{x}) = 0$$





Note: if expanding one can take infinite-volume QED

but putting QED and QCD in different spaces causes its own subtleties

- figures from A. Nicholson, GHP 2021 •

That’s all… thanks for listening!