Exercises for
"QCD (and QED) in a Euclidean Finite-Volume"
Days 2 and 3, EuroPLEx 2021

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1 Old-fashioned perturbation theory

Consider the diagram shown in figure 1. Write out the expression for the diagram using $-ig$ for the coupling and

$$D(p) = \frac{1}{i}\frac{1}{p^2 + m^2 - i\epsilon}\,, \tag{1}$$

for the propagator. (The external propagators here are amputated.)

Evaluate the temporal component and express the result as the sum of two terms, one depending on $1/(E - 2\omega_{\boldsymbol{k}})$ and the other on $1/(E + 2\omega_{\boldsymbol{k}})$.

Can you devise a diagrammatic rule that leads to these two terms individually, based on time ordering the vertices?

Can you think how to extend this to all diagrams and prove that it is equivalent to usual perturbation theory?

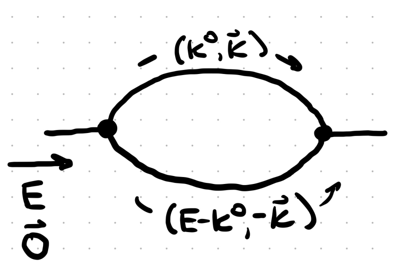

Figure 1: Diagram for exercise 1.

2 $1/L$ expansion of the ground-state energy

Take the two-particle quantization condition in the form

$$p \cot \delta(p) = -\frac{1}{a_0} = \frac{1}{\pi L}\left[\sum_{\boldsymbol{n}} - \text{p.v.} \int d^3\boldsymbol{n}\right] \frac{1}{\boldsymbol{n}^2 - q^2}\,, \tag{2}$$

where $E = 2\sqrt{m^2 + p^2}$ and $q = Lp/(2\pi)$. Here a_0 is the two-particle scattering length.

2.1 Expanding the finite-volume function

First consider a cut-off version of the integral

$$\text{p.v.} \int^{\Lambda} d^3\boldsymbol{n} \frac{1}{\boldsymbol{n}^2 - q^2} = \text{p.v.} \int^{\Lambda} d^3\boldsymbol{n} \frac{1}{\boldsymbol{n}^2} + \text{p.v.} \int^{\Lambda} d^3\boldsymbol{n} \left[\frac{1}{\boldsymbol{n}^2 - q^2} - \frac{1}{\boldsymbol{n}^2}\right]. \tag{3}$$

Prove that the second integral vanishes as $\Lambda \to \infty$ and evaluate the first. Now write

$$\lim_{\Lambda \to \infty} \left[\sum_{\boldsymbol{n}}^{\Lambda} - \text{p.v.} \int^{\Lambda} d^3\boldsymbol{n}\right] \frac{1}{\boldsymbol{n}^2 - q^2} = -\frac{1}{q^2} + \mathcal{C} \times (q^2)^0 + \mathcal{O}(q^2)\,, \tag{4}$$

and determine the value of $\mathcal{C}$.

2.2 Expanding the energy

Use these results to determine $E_0(L)$ through order $1/L^4$.

3 Non-interacting Lellouch-Lüscher factor

Recall our relation for $K \to \pi\pi$

$$|\langle \pi\pi, \text{out}|\mathcal{H}_W(0)|K\rangle|^2 = 2M_K \mathcal{R}^{-1} |\langle n, L|H_W(0)|K\rangle|^2\,, \tag{5}$$

with

$$\mathcal{R}(E_n(L), L)^{-1} = \partial_E \left[F_{i\epsilon}(E, L)^{-1} + \mathcal{M}(E)\right]\Big|_{E = E_n(L)}\,. \tag{6}$$

3.1 Non-interacting set-up

Give the expression for $\mathcal{R}$ and $E_n(L)$ in the non-interacting limit. (This is intended to take two lines.)

3.2 Evaluating non-interacting $\mathcal{R}$

In the non-interacting limit the integral part of $F_{i\epsilon}$ does not contribute and we can write

$$F_{i\epsilon}(E, L) = \frac{1}{2}\frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{1}{(2\omega_{\boldsymbol{k}})^2(E - 2\omega_{\boldsymbol{k}})} \,. \tag{7}$$

Use this to evaluate $\mathcal{R}^{-1}$ and write a simplified version of equation (5). Explain why the result is expected.