Feynman Integrals

Lecture 1

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Section 1

Organisation

Organisation

- Online course on Feynman integrals via zoom.
- Daily schedule for this course:

9:00-11:00	**Lecture** followed by a **discussion session** in the plenum
11:30-12:30	**Tutorial**

- Slides and exercise sheets on the indico page.
- Afternoon: Lectures by Matthew McCullough

Tutorials

- For each lecture there will be an exercise sheet.
- Tutorials: Small groups (5-7 people) in breakout rooms
- Tutor: Yao Ma
- Close-out at 12:20 in the plenum.

Advice

- A summer school has a scientific part and a social part.
- **Talk** to your fellow students in the tutorials.
- **Ask** questions in the discussion session.
- It's your summer school!

Section 2

Motivation

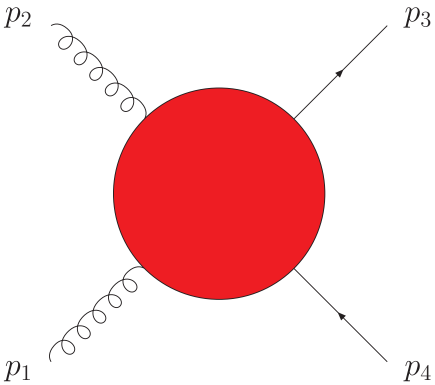
Scattering amplitudes

- We would like to make precise predictions for observables in scattering experiments from quantum field theory.
- Any such calculation will involve a scattering amplitude.
- Unfortunately we cannot calculate scattering amplitudes exactly.
- If we have a small parameter like a small coupling, we may use perturbation theory.
- We may organise the perturbative expansion of a scattering amplitude in terms of Feynman diagrams.

Scattering amplitude = sum of all Feynman diagrams

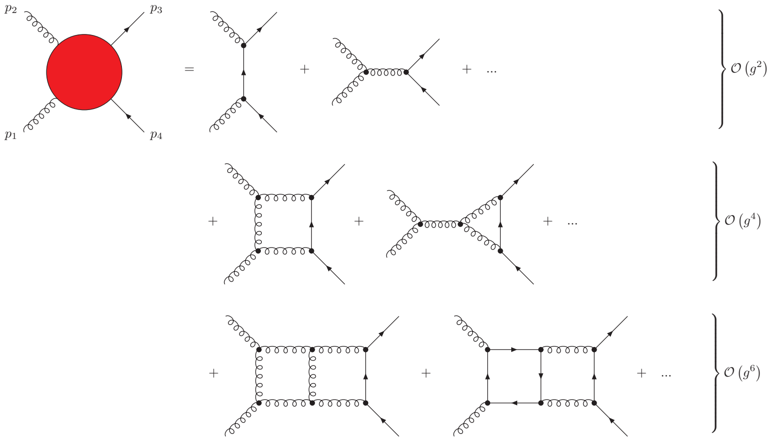
Scattering amplitudes

Example: **Scattering amplitude** for a $2 \to 2$-process:



Scattering amplitudes

We may compute the scattering amplitude within perturbation theory:



Feynman diagrams

- At leading order we (usually) have only tree diagrams. These do not pose any conceptual problem.
- But precise predictions require more terms in the perturbative expansion.
- Beyond leading order we have loop diagrams and we have to compute loop integrals.
- Content of this course: **How do we compute loop integrals?**

Section 3

Basics

Introducing Feynman integrals

We may define Feynman integrals without knowledge of quantum field theory.
We just need:

- Special relativity
- Graphs

Special relativity

- Denote by D the **number of space-time dimensions**.

  In our real world D equals 4 (one time dimension and three spatial dimensions), but it is extremely helpful to keep this number arbitrary. We will always assume that space-time consists of one time dimension and $(D-1)$ spatial dimensions.

- The **momentum of a particle** is a D-dimensional vector, whose first component gives the energy E (divided by the speed of light c) and the remaining $(D-1)$ components give the components of the spatial momentum, which we label with superscripts:

$$p = \left(\frac{E}{c}, p^1, \dots, p^{D-1} \right).$$

- It is common practice to work in **natural units**, where

$$c = \hbar = 1.$$

Special relativity

- In natural units we write $p^0 = E$ and

$$p = \left(p^0, p^1, \dots, p^{D-1}\right).$$

A component of p is denoted by

$$p^\mu \quad \text{with} \quad 0 \leq \mu \leq D-1$$

The index μ is called a **Lorentz index**.

- **Minkowski metric**: $g_{\mu\nu} = \mathrm{diag}(1, -1, \dots, -1)$.
- **Minkowski scalar product** of two momentum vectors p_a and p_b:

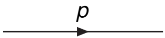
$$p_a \cdot p_b = \sum_{\mu=0}^{D-1} \sum_{\nu=0}^{D-1} p_a^\mu \, g_{\mu\nu} \, p_b^\nu = p_a^\mu \, g_{\mu\nu} \, p_b^\nu.$$

In the last expression we used Einstein's summation convention.

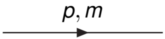
Special relativity

We represent a particle propagating in space-time by a line:

If we would like to indicate the direction of the momentum flow, we optionally put an arrow:



The line is our **first building block** for a Feynman graph. A line for a particle with momentum p and mass m stands for



$$= \frac{1}{-p^2 + m^2}.$$

Special relativity

Remark: If one follows standard conventions in quantum field theory the propagator of a scalar particle with momentum p and mass m is given by

$$\frac{i}{p^2 - m^2}.$$

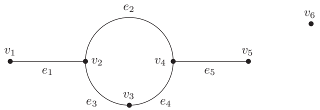
This differs by a factor $(-i)$ from

$$\frac{1}{-p^2 + m^2}.$$

This is just a prefactor and easily adjusted.

Graphs

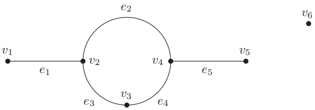
- An **unoriented graph** consists of edges and vertices, where an edge connects two vertices.
- A graph may be connected or disconnected. We will mainly consider **connected graphs**.
- An **oriented graph** is a graph, where for every edge an orientation is chosen. An orientated edge is usually drawn with an arrow line:



Valency of a vertex

The **valency of a vertex** is the number of edges attached to it.

- A vertex of valency 0 is necessarily disconnected from the rest of graph.
- A vertex of valency 1 has exactly one edge attached to it. This edge is called an **external edge**. All other edges are called **internal edges**.
- A vertex of valency 2 is also called a **dot**.



Momentum conservation

Choose an orientation for every edge.

$$\begin{aligned} E^{\mathrm{source}}(v_a) & : \quad \text{set of edges, which have vertex } v_a \text{ as source,} \\ E^{\mathrm{sink}}(v_a) & : \quad \text{set of edges, which have vertex } v_a \text{ as sink.} \end{aligned}$$

At each vertex v_a of valency > 1 we impose **momentum conservation**:

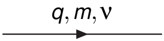
$$\sum_{e_j \in E^{\mathrm{source}}(v_a)} q_j = \sum_{e_j \in E^{\mathrm{sink}}(v_a)} q_j .$$

Dots

Consider a vertex of valency 2 inside a Feynman graph:

$$= \frac{1}{\left(-q^2+m^2\right)^2}.$$

We get the same effect if we associate to each edge in addition to p and m a number ν, corresponding to the power of the propagator:



$$= \frac{1}{\left(-q^2+m^2\right)^{\nu}}$$

The loop number

Consider a graph G with n edges, r vertices and k connected components. The **loop number** l is defined by

$$l = n - r + k.$$

If the graph is connected we have

$$l = n - r + 1.$$

The loop number l is also called the **first Betti number** of the graph or the **cyclomatic number**.

Physics interpretation: If we fix all momenta of the external lines and if we impose momentum conservation at each vertex, then the loop number is equal to the number of independent momentum vectors not constrained by momentum conservation.

Trees and forests

- A connected graph of loop number 0 is called a **tree**.
- A graph of loop number 0, connected or not, is called a **forest**.
  If the forest has k connected components, it is called a k-forest.
  A tree is a 1-forest.

Feynman graphs which are trees pose no conceptual problem.

Our focus in this lecture is on connected Feynman graphs, which are not trees, e.g. Feynman graphs with loop number $l > 0$.

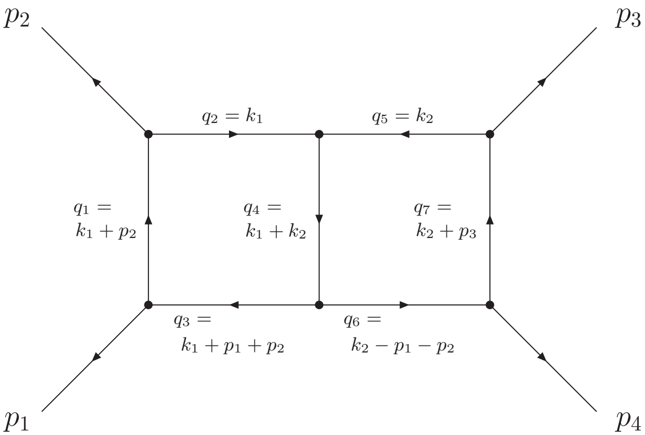
Notation

Consider a connected graph G with n_{ext} external edges, n_{int} internal edges and loop number l:

External momenta: $p_1, \dots, p_{n_{\text{ext}}}$
Internal momenta: $q_1, \dots, q_{n_{\text{int}}}$
Independent loop momenta: $k_1, \dots, k_l$

$$q_j = \sum_{r=1}^{l} \lambda_{jr} k_r + \sum_{r=1}^{n_{\text{ext}}-1} \sigma_{jr} p_r, \qquad \lambda_{jr}, \sigma_{jr} \in \{-1, 0, 1\}.$$

Example



p_2
p_3
$q_2 = k_1$
$q_5 = k_2$
$q_1 =$
$k_1 + p_2$
$q_4 =$
$k_1 + k_2$
$q_7 =$
$k_2 + p_3$
$q_3 =$
$k_1 + p_1 + p_2$
$q_6 =$
$k_2 - p_1 - p_2$
p_1
p_4

Quiz

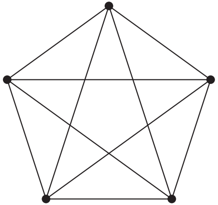
How many loops does this graph have?

(A) 5

(B) 6

(C) 11

(D) 16



Integration

- We may consider the external momenta as input data, but what shall we do with the l independent loop momenta?
- Quantum field theory instructs us to **integrate over the independent loop momenta**. Thus we include for every independent loop momentum k_r a D-dimensional integration

$$\int \frac{d^D k_r}{i\pi^{\frac{D}{2}}}.$$

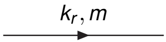
- If one follows standard conventions in quantum field theory, the measure is given by

$$\int \frac{d^D k_r}{(2\pi)^D}.$$

The difference is a simple prefactor and one easily converts from one convention to the other convention.

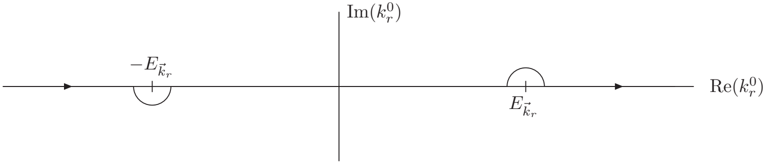
Integration domain

- Naive expectation: We integrate each of the D components of k_r along the real axis from $-\infty$ to $+\infty$.
- However, there will be poles on the real axis:

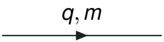


$$= \frac{1}{-k_r^2 + m^2}$$

- Quantum field theory dictates us that the correct **integration contour** is:



- Feynman's $i\delta$-prescription:



$$= \frac{1}{-q^2 + m^2 - i\delta}$$

Notation

- For the space-time dimension D we write

$$D = 4 - 2\varepsilon.$$

- Euler's constant γ_E:

$$\gamma_E = \lim_{n\to\infty}\left(-\ln n + \sum_{j=1}^{n}\frac{1}{j}\right) = 0.57721566490153286\dots$$

- In order to enforce that our Feynman integral is dimensionless, we introduce an arbitrary parameter μ with mass dimension $\dim(\mu) = 1$ and multiply by an appropriate power of μ.

The Feynman integral

Definition

The Feynman integral for a Feynman graph G with n_{ext} external edges, n_{int} internal edges and l loops is given in D space-time dimensions by

$$I = e^{l\varepsilon\gamma_{\mathrm{E}}} \left(\mu^2\right)^{\nu - \frac{lD}{2}} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}},$$

where each internal edge e_j of the graph is associated with a triple (q_j, m_j, ν_j),

$$q_j = \sum_{r=1}^{l} \lambda_{jr} k_r + \sum_{r=1}^{n_{\mathrm{ext}}-1} \sigma_{jr} p_r, \qquad \nu = \sum_{j=1}^{n_{\mathrm{int}}} \nu_j.$$

The coefficients λ_{jr} and σ_{jr} can be obtained from momentum conservation at each vertex of valency > 1.

The integration contour is given by Feynman's $i\delta$-prescription.

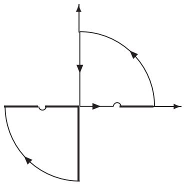
Wick rotation

Minkowski scalar product:

$$k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2 - \dots$$

If the quarter-circles at infinity give no contribution:

$$\oint dk_0 f(k_0) = 0 \quad \Rightarrow \quad \int\limits_{-\infty}^{\infty} \mathbf{dk_0 f(k_0)} = -\int\limits_{i\infty}^{-i\infty} \mathbf{dk_0 f(k_0)}$$



Change variables:

$$\begin{aligned} k_0 &= iK_0, \\ k_j &= K_j, \quad \text{for } 1 \leq j \leq D-1. \end{aligned}$$

Final result:

$$\int \frac{d^D k}{i\pi^{D/2}} f(-k^2) = \int \frac{d^D K}{\pi^{D/2}} f(K^2),$$

with Euclidean scalar product $K^2 = K_0^2 + K_1^2 + K_2^2 + K_3^2 + \dots$.

The need for regularisation

Loop integrals in four space-time dimensions can be divergent!

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} = \frac{1}{(4\pi)^2} \int\limits_0^\infty dk^2 \frac{1}{k^2} = \frac{1}{(4\pi)^2} \int\limits_0^\infty \frac{dx}{x}$$

This integral diverges at

- $k^2 \to \infty$ (**UV-divergence**) and at
- $k^2 \to 0$ (**IR-divergence**).

Dimensional regularisation

- Replace the four-dimensional integral by an D-dimensional integral, where D is now an additional parameter, which can be a non-integer or even a complex number.
- We consider the result of the integration as a function of D and we are interested in the behaviour of this function as D approaches 4.
- Concept closely related to: Consider a function $f(z)$, which is defined for any positive integer $n \in \mathbb{N}$ by

$$f(n) = n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n.$$

We would like to define $f(z)$ for any value $z \in \mathbb{C}$ (except for a countable set of isolated points, where $f(z)$ is allowed to have poles).
Answer is well known and given by Euler's gamma function

$$f(z) = \Gamma(z+1).$$

Dimensional regularisation

- A Feynman integral I has a **Laurent expansion** in ε:

$$I = \sum_{j=j_{\min}}^{\infty} \varepsilon^j I^{(j)},$$

where $I^{(j)}$ denotes the coefficient of ε^j.

- For precision calculations we are interested in the first few terms $I^{(j_{\min})}$, $I^{(j_{\min}+1)}$, ... of this Laurent series.
The exact number of required terms depends on the order of perturbation theory we are calculating.
- Let us stress that we would like to get the $I^{(j)}$'s, not necessarily I itself.
There are situations where a closed form expression for I is readily obtained, but the Laurent expansion in ε is not immediate.

The tadpole integral

Let's get our hands dirty: We start with a one-loop integral, which only depends on the loop momentum squared (and no scalar product $k \cdot p$ with some external momentum).

The **one-loop tadpole integral**:

$$T_\nu = e^{\varepsilon \gamma_E} \left(\mu^2\right)^{\nu - \frac{D}{2}} \int \frac{d^D k}{i \pi^{\frac{D}{2}}} \frac{1}{\left(-k^2 + m^2\right)^\nu}.$$

After Wick rotation we have

$$T_\nu = e^{\varepsilon \gamma_E} \left(\mu^2\right)^{\nu - \frac{D}{2}} \int \frac{d^D K}{\pi^{\frac{D}{2}}} \frac{1}{\left(K^2 + m^2\right)^\nu}.$$

Spherical coordinates

One radial variable K, $D-2$ polar angles θ_1, ..., θ_{D-2} and one azimuthal angle θ_{D-1}:

$$
\begin{aligned}
K_0 &= K \cos\theta_1 \\
K_1 &= K \sin\theta_1 \cos\theta_2 \\
&\;\;... \\
K_{D-2} &= K \sin\theta_1 ... \sin\theta_{D-2} \cos\theta_{D-1} \\
K_{D-1} &= K \sin\theta_1 ... \sin\theta_{D-2} \sin\theta_{D-1}
\end{aligned}
$$

The measure becomes

$$
d^D K = K^{D-1} dK d\Omega_D
$$

Integration over the angles yields

$$
\int d\Omega_D = \int\limits_0^{\pi} d\theta_1 \sin^{D-2}\theta_1 ... \int\limits_0^{\pi} d\theta_{D-2} \sin\theta_{D-2} \int\limits_0^{2\pi} d\theta_{D-1} = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}
$$

Euler's gamma function

The **gamma function** is defined for $\mathrm{Re}(z) > 0$ by

$$\Gamma(z) = \int\limits_0^\infty e^{-t} t^{z-1} dt.$$

It fulfils the functional equation

$$\Gamma(z+1) = z\,\Gamma(z).$$

For positive integers n it takes the values

$$\Gamma(n+1) = n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n.$$

The expansion around $z = 1$ reads

$$\Gamma(1+\varepsilon) = \exp\left(-\gamma_{\mathrm{E}}\varepsilon + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_n \varepsilon^n\right), \qquad \zeta_n = \sum_{j=1}^{\infty} \frac{1}{j^n}.$$

ζ_n is called a **zeta value**. Laurent expansion around $z = 0$:

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_{\mathrm{E}} + O(\varepsilon).$$

Euler's beta function

Euler's beta function is defined for $\mathrm{Re}(z_1) > 0$ and $\mathrm{Re}(z_2) > 0$ by

$$B(z_1, z_2) = \int_0^1 t^{z_1 - 1} (1 - t)^{z_2 - 1} dt,$$

or equivalently by

$$B(z_1, z_2) = \int_0^\infty \frac{t^{z_1 - 1}}{(1 + t)^{z_1 + z_2}} dt.$$

The beta function can be expressed in terms of Gamma functions:

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}.$$

The tadpole integral

Performing the angular integrations for our tadpole integral we obtain

$$T_\nu = \frac{e^{\varepsilon\gamma_{\mathrm{E}}} \left(\mu^2\right)^{\nu-\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int\limits_0^\infty dK^2 \frac{\left(K^2\right)^{\frac{D}{2}-1}}{\left(K^2+m^2\right)^\nu}.$$

We substitute $t = K^2/m^2$ and obtain

$$T_\nu = \frac{e^{\varepsilon\gamma_{\mathrm{E}}}}{\Gamma\left(\frac{D}{2}\right)} \left(\frac{m^2}{\mu^2}\right)^{\frac{D}{2}-\nu} \int\limits_0^\infty dt \frac{t^{\frac{D}{2}-1}}{(t+1)^\nu}.$$

The remaining integral is just Euler's beta function

$$\int\limits_0^\infty dt \frac{t^{\frac{D}{2}-1}}{(1+t)^\nu} = \frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\nu-\frac{D}{2}\right)}{\Gamma(\nu)}.$$

The tadpole integral

Tadpole integral*:*

$$T_\nu\left(D, \frac{m^2}{\mu^2}\right) = \frac{e^{\varepsilon\gamma_E}\Gamma\left(\nu - \frac{D}{2}\right)}{\Gamma(\nu)}\left(\frac{m^2}{\mu^2}\right)^{\frac{D}{2}-\nu}.$$

We are interested in the Laurent expansion of Feynman integrals. With $D = 4 - 2\varepsilon$, $\nu = 1$ and $L = \ln(m^2/\mu^2)$ we obtain

$$\begin{aligned} T_1(4-2\varepsilon) &= \frac{m^2}{\mu^2} e^{\varepsilon\gamma_E}\Gamma(-1+\varepsilon)\, e^{-\varepsilon L} \\ &= \frac{m^2}{\mu^2}\left[-\frac{1}{\varepsilon} + (L-1) + \left(-\frac{1}{2}L^2 - \frac{1}{2}\zeta_2 + L - 1\right)\varepsilon\right] + \mathcal{O}\left(\varepsilon^2\right). \end{aligned}$$

The pole at $\varepsilon = 0$ originates from the ultraviolet divergence of tadpole integral with $\nu = 1$ in four-space time dimensions.

The tadpole integral

Tadpole integral:

$$T_\nu\left(D, \frac{m^2}{\mu^2}\right) = \frac{e^{\varepsilon\gamma_E}\Gamma\left(\nu - \frac{D}{2}\right)}{\Gamma(\nu)}\left(\frac{m^2}{\mu^2}\right)^{\frac{D}{2}-\nu}.$$

The result is valid for any D, so we may as well expand it around two space-time dimensions. Just for fun, let's do it:

$$\begin{aligned} \varepsilon\, T_1(2-2\varepsilon) &= e^{\varepsilon\gamma_E}\Gamma(1+\varepsilon)\, e^{-\varepsilon L} \\ &= 1 - L\varepsilon + \left(\frac{1}{2}L^2 + \frac{1}{2}\zeta_2\right)\varepsilon^2 + O\left(\varepsilon^3\right). \end{aligned}$$

Weights

Define a weight as follows:

- Any algebraic expression in the kinematic variables has weight 0.
- The dimensional regularisation parameter has weight (-1).
- $L = \ln(m^2/\mu^2)$ has weigth 1, ζ_n has weight n.
- The weight of a product is the sum of the weights of its factors.

We observe

- $T_1(4-2\varepsilon)$ contains terms of mixed weight.
- $\varepsilon\, T_1(2-2\varepsilon)$ only involves terms of weight zero.

We call $\varepsilon\, T_1(2-2\varepsilon)$ to be of **uniform weight**.

Dimensional shift relations

We may show

$$T_\nu(D) = \nu T_{\nu+1}(D+2).$$

This is an example of a **dimensional shift relation**, relating integrals in D and $D+2$ space-time dimensions.

Master integrals

Write $D = [D] - 2\varepsilon$ and set

$$J_1 \quad = \quad \varepsilon\, T_1\,(2 - 2\varepsilon) \;=\; e^{\varepsilon \gamma_{\mathrm{E}}} \Gamma\,(1+\varepsilon)\, e^{-\varepsilon L}.$$

It is not too difficult to show that

$$T_\nu\,(D) \quad = \quad \frac{\Gamma\left(\nu - \frac{[D]}{2} + \varepsilon\right)}{\Gamma\,(\nu)\,\Gamma\,(1+\varepsilon)} \left(\frac{m^2}{\mu^2}\right)^{\left(\frac{[D]}{2} - \nu\right)} J_1\,.$$

For $\nu \in \mathbb{N}$ and $[D]$ even, the prefactor is always a rational function in ε and m^2. Thus we may express $T_\nu(D)$ as a rational coefficient times J_1.

Later on we will see, that this generalises as follows: We may express any member of a family of Feynman integrals as a linear combination of Feynman integrals from a finite set. The Feynman integrals from this finite set are called **master integrals**. A master integrals, which is of uniform weight, is called a **canonical master integral**.

Thus J_1 is a canonical master integral.