Solutions for the course “Feynman integrals”
Sheet 1

Exercise 1

Consider a connected graph G with n_{ext} external edges, n_{int} internal edges and loop number l. Denote by

$$\begin{aligned} E^{\text{in}} &: \text{ set of edges, which have a vertex of valency 1 as source,} \\ E^{\text{out}} &: \text{ set of edges, which have a vertex of valency 1 as sink.} \end{aligned}$$

The edges in E^{in} and E^{out} are necessarily external edges.

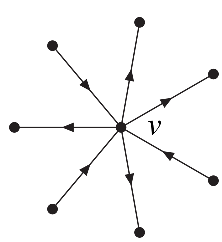
Show that momentum conservation at each vertex of valency > 1 implies momentum conservation of the external momenta:

$$\sum_{e_j \in E^{\text{in}}} q_j = \sum_{e_j \in E^{\text{out}}} q_j.$$

In other words: Momentum conservation at each internal vertex implies momentum conservation of the external momenta.

Solution:

We proof the claim by induction on the number of internal edges n_{int}. For $n_{\text{int}} = 0$ our graph looks like



and has exactly one vertex v of valency > 1. Momentum conservation at this vertex reads

$$\sum_{e_j \in E^{\text{in}}} q_j = \sum_{e_j \in E^{\text{out}}} q_j.$$

and corresponds exactly to the claim.

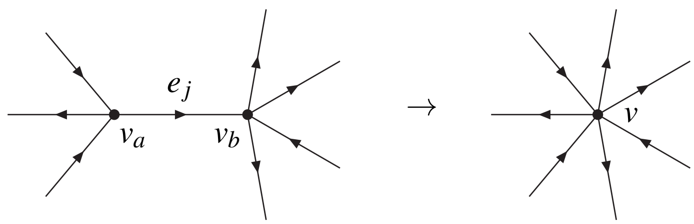
Let us now assume that the claim is correct for $(n_{\text{int}} - 1)$. Consider now a graph with n_{int} internal edges. Pick one internal edge e_i. Denote by v_a its source and by v_b its sink. Let us write down momentum conservation at v_a and v_b:

$$\begin{aligned} v_a: \;\; & q_j + \sum_{e_r \in E^{\text{source}}(v_a) \backslash \{e_j\}} q_r = \sum_{e_r \in E^{\text{sink}}(v_a)} q_r, \\ v_b: \;\; & \sum_{e_r \in E^{\text{source}}(v_b)} q_r = q_j + \sum_{e_r \in E^{\text{sink}}(v_b) \backslash \{e_j\}} q_r. \end{aligned} \tag{1}$$

We may eliminate q_i from these two equations. We obtain a single equation

$$v: \sum_{e_r \in E^{\text{source}}(v_a) \backslash \{e_j\}} q_r + \sum_{e_r \in E^{\text{source}}(v_b)} q_r = \sum_{e_r \in E^{\text{sink}}(v_a)} q_r + \sum_{e_r \in E^{\text{sink}}(v_b) \backslash \{e_j\}} q_r. \qquad (2)$$

The momentum q_j appears at no other vertex. As far as momentum conservation is concerend, we may replace the graph G with a new graph $\tilde{G}$, where the edge e_j has been contracted (e.g. the edge e_j is removed and the vertices v_a and v_b are merged to a new vertex v. Pictorially we have



The new graph $\tilde{G}$ has $(n_{\text{int}} - 1)$ internal edges and we may use the induction hypothesis. As G and $\tilde{G}$ only differ by the contraction of an internal edge, the claim holds for the graph G as well.

If we choose an orientation such that all external edges have a vertex of valency 1 as sink we have

$$E^{\text{in}} = \emptyset, \qquad E^{\text{out}} = \{e_{n_{\text{int}}+1}, \dots, e_{n_{\text{int}}+n_{\text{ext}}}\}$$

and

$$0 = \sum_{e_j \in E^{\text{in}}} q_j = \sum_{e_j \in E^{\text{out}}} q_j = \sum_{j=1}^{n_{\text{ext}}} p_j.$$

Exercise 2

Let $U, V > 0$ be some constants and $a, \nu \in \mathbb{C}$. Show that

$$\int \frac{d^D k}{i\pi^{\frac{D}{2}}} \frac{\left(-k^2\right)^a}{\left(-Uk^2+V\right)^\nu} = \frac{\Gamma\left(\frac{D}{2}+a\right)}{\Gamma\left(\frac{D}{2}\right)} \frac{\Gamma\left(\nu - \frac{D}{2} - a\right)}{\Gamma(\nu)} \frac{U^{-\frac{D}{2}-a}}{V^{\nu-\frac{D}{2}-a}}.$$

Hint: Repeat the steps we did in the calculation of the tadpole integral.

Solution:

Denote

$$\tilde{T} = e^{\varepsilon \gamma_E} \left(\mu^2\right)^{\nu - \frac{D}{2} - a} \int \frac{d^D k}{i\pi^{\frac{D}{2}}} \frac{\left(-k^2\right)^a}{\left(-Uk^2+V\right)^\nu}.$$

We repeat the steps from the calculation of the tadpole integral. Starting from

$$\tilde{T} = e^{\varepsilon\gamma_E} \left(\mu^2\right)^{\nu-\frac{D}{2}-a} \int \frac{d^D k}{i\pi^{\frac{D}{2}}} \frac{\left(-k^2\right)^a}{\left(-Uk^2+F\right)^\nu}$$

we perform a Wick rotation and obtain

$$\tilde{T} = e^{\varepsilon\gamma_E} \left(\mu^2\right)^{\nu-\frac{D}{2}-a} \int \frac{d^D K}{\pi^{\frac{D}{2}}} \frac{\left(K^2\right)^a}{\left(UK^2+F\right)^\nu}.$$

We then introduce spherical coordinates and integrate over the angles. This yields

$$\tilde{T} = \frac{e^{\varepsilon\gamma_E} \left(\mu^2\right)^{\nu-\frac{D}{2}-a}}{\Gamma\left(\frac{D}{2}\right)} \int\limits_0^\infty dK^2 \frac{\left(K^2\right)^{\frac{D}{2}+a-1}}{\left(UK^2+F\right)^\nu}.$$

We then substitute $t = UK^2/F$. We obtain

$$\tilde{T} = \frac{e^{\varepsilon\gamma_E} \left(\mu^2\right)^{\nu-\frac{D}{2}-a}}{\Gamma\left(\frac{D}{2}\right)} \frac{U^{-\frac{D}{2}-a}}{F^{\nu-\frac{D}{2}-a}} \int\limits_0^\infty dt \frac{t^{\frac{D}{2}+a-1}}{\left(t+1\right)^\nu}.$$

The remaining integral is again Euler's beta integral and we finally obtain

$$\tilde{T} = e^{\varepsilon\gamma_E} \left(\mu^2\right)^{\nu-\frac{D}{2}-a} \frac{\Gamma\left(\frac{D}{2}+a\right)}{\Gamma\left(\frac{D}{2}\right)} \frac{\Gamma\left(\nu-\frac{D}{2}-a\right)}{\Gamma(\nu)} \frac{U^{-\frac{D}{2}-a}}{F^{\nu-\frac{D}{2}-a}}.$$