Feynman Integrals

Lecture 2

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Section 1

Representations of Feynman integrals

Subsection 1

The momentum representation

The momentum representation

The Feynman integral for a Feynman graph G with n_{ext} external edges, n_{int} internal edges and l loops is given in D space-time dimensions by

$$I = e^{l\varepsilon\gamma_{\mathrm{E}}} \left(\mu^2\right)^{\nu - \frac{lD}{2}} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}},$$

where each internal edge e_j of the graph is associated with a triple (q_j, m_j, ν_j),

$$q_j = \sum_{r=1}^{l} \lambda_{jr} k_r + \sum_{r=1}^{n_{\mathrm{ext}}-1} \sigma_{jr} p_r, \qquad \nu = \sum_{j=1}^{n_{\mathrm{int}}} \nu_j.$$

The coefficients λ_{jr} and σ_{jr} can be obtained from momentum conservation at each vertex of valency > 1.

Variables

The Feynman integral depends on:

- The **dimension of space-time** $D \in \mathbb{C}$ (or more precisely on $D_{\text{int}} \in \mathbb{N}$ and $\varepsilon \in \mathbb{C}$).
- The **exponents of the propagators** $(\nu_1, \dots, \nu_{n_{\text{int}}})$. In principle we may allow $\nu_j \in \mathbb{C}$, but very often we will limit us to the case $\nu_j \in \mathbb{Z}$.
- **Kinematic variables**:
  - A scalar Feynman integral depends on the external momenta only through the Lorentz invariants $p_i \cdot p_j$.
  - A dimensionless Feynman integral depends on the Lorentz invariants, the internal masses and the scale μ only through the dimensionless ratios

$$\frac{-p_i \cdot p_j}{\mu^2}, \qquad \frac{m_i^2}{\mu^2}.$$

We denote the dimensionless kinematic variables by $x_1, x_2, \dots$.

Kinematic variables

How many kinematic variables can we have maximally?

- $n_{\text{ext}}(n_{\text{ext}}-1)/2$ kinematic variables of the type $(-p_i \cdot p_j)/\mu^2$.
- n_{int} kinematic variables of the type m_i^2/μ^2.
- **Scaling relation**: If we rescale all kinematic variables by a factor λ we have

$$I(\lambda x_1, \lambda x_2, \dots) = \lambda^{\frac{lD}{2}-\nu} I(x_1, x_2, \dots).$$

  Allows us to set one kinematic variable to one.

Thus the maximal number of kinematic variables is

$$\frac{n_{\text{ext}}(n_{\text{ext}}-1)}{2} + n_{\text{int}} - 1.$$

Very often some of them are zero (for example some internal masses might be zero) or identical (for example some internal masses might be identical).

Kinematic variables

Notation:

number of independent kinematic variables:	N_B
independent kinematic variables:	$x_1, x_2, \dots, x_{N_B}$
Feynman integral:	$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B})$

Euclidean region: All x_j's are real and non-negative (i.e. $x_j \geq 0$)

Subsection 2

The Schwinger parameter representation

Schwinger parameters

Let $A > 0$ and $\mathrm{Re}(\nu) > 0$. We have

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int\limits_0^\infty d\alpha \, \alpha^{\nu-1} \, e^{-\alpha A}$$

and therefore

$$\frac{1}{(-q_j^2 + m_j^2)^{\nu_j}} = \frac{1}{\Gamma(\nu_j)} \int\limits_{\alpha_j \geq 0} d\alpha_j \, \alpha_j^{\nu_j - 1} \exp\left(-\alpha_j(-q_j^2 + m_j^2)\right)$$

The variable α_j is called a **Schwinger parameter**.

Let's apply this to all propagators:

$$I = \frac{e^{l\varepsilon\gamma_{\mathrm{E}}}\left(\mu^2\right)^{\nu-\frac{lD}{2}}}{\prod\limits_{j=1}^{n_{\mathrm{int}}}\Gamma(\nu_j)} \int\limits_{\alpha_j\geq 0} d^{n_{\mathrm{int}}}\alpha \left(\prod_{j=1}^{n_{\mathrm{int}}}\alpha_j^{\nu_j-1}\right) \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \exp\left(-\sum_{j=1}^{n_{\mathrm{int}}}\alpha_j\left(-q_j^2+m_j^2\right)\right)$$

Using $q_j = \sum\limits_{r=1}^{l}\lambda_{jr}k_r + \sum\limits_{r=1}^{n_{\mathrm{ext}}-1}\sigma_{jr}p_r$ we express the argument of the exponential function as

$$\sum_{j=1}^{n_{\mathrm{int}}}\alpha_j(-q_j^2+m_j^2) = -\sum_{r=1}^{l}\sum_{s=1}^{l}k_r M_{rs} k_s + \sum_{r=1}^{l} 2k_r\cdot v_r + J$$

M : *$l\times l$ matrix with scalar entries*

v : *l-vector with D-dimensional momentum vectors as entries*

J : *scalar*

Graph polynomials

$$\mathcal{U} = det(M) \qquad \mathcal{F} = det(M)\left(J + v^T M^{-1} v\right)/\mu^2$$

The functions $\mathcal{U}$ and $\mathcal{F}$ are called **graph polynomials** (they are polynomials in α_j's).

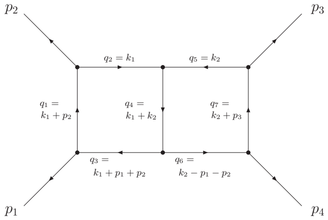
- $\mathcal{U}$ and $\mathcal{F}$ are homogeneous in the Schwinger parameters, $\mathcal{U}$ is of degree l, $\mathcal{F}$ is of degree $l+1$.
- $\mathcal{U}$ is linear in each Schwinger parameter. If all internal masses are zero, then also $\mathcal{F}$ is linear in each Schwinger parameter.
- In expanded form each monomial of $\mathcal{U}$ has coefficient $+1$.

We call $\mathcal{U}$ the **first Symanzik polynomial** and $\mathcal{F}$ the **second Symanzik polynomial**.

Example

Consider the two-loop double box graph for the case

$$p_1^2 = 0, \quad p_2^2 = 0, \quad p_3^2 = 0, \quad p_4^2 = 0,$$
$$m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = 0.$$



Define the Mandelstam variables

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 , \\ t &= (p_2 + p_3)^2 = (p_1 + p_4)^2 . \end{aligned}$$

Example

$$\sum_{j=1}^{7} \alpha_j \left(-q_j^2\right) = -\left(\alpha_1+\alpha_2+\alpha_3+\alpha_4\right) k_1^2 - 2\alpha_4 k_1 \cdot k_2 - \left(\alpha_4+\alpha_5+\alpha_6+\alpha_7\right) k_2^2$$
$$+2\left[\alpha_1 p_1 + \alpha_2 \left(p_1+p_2\right)\right] \cdot k_1 + 2\left[\alpha_5 \left(p_3+p_4\right) + \alpha_7 p_4\right] \cdot k_2 - \left(\alpha_2+\alpha_5\right) s.$$

Extract M, v, J:

$$M = \begin{pmatrix} \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \alpha_4 \\ \alpha_4 & \alpha_4+\alpha_5+\alpha_6+\alpha_7 \end{pmatrix},$$
$$v = \begin{pmatrix} \alpha_1 p_1 + \alpha_2 \left(p_1+p_2\right) \\ \alpha_5 \left(p_3+p_4\right) + \alpha_7 p_4 \end{pmatrix},$$
$$J = \left(\alpha_2+\alpha_5\right)\left(-s\right).$$

We obtain the graph polynomials as

$$\mathcal{U} = \left(\alpha_1+\alpha_2+\alpha_3\right)\left(\alpha_5+\alpha_6+\alpha_7\right) + \alpha_4 \left(\alpha_1+\alpha_2+\alpha_3+\alpha_5+\alpha_6+\alpha_7\right),$$
$$\mathcal{F} = \left[\alpha_2\alpha_3\left(\alpha_4+\alpha_5+\alpha_6+\alpha_7\right) + \alpha_5\alpha_6\left(\alpha_1+\alpha_2+\alpha_3+\alpha_4\right) + \alpha_2\alpha_4\alpha_6 + \alpha_3\alpha_4\alpha_5\right] \left(\frac{-s}{\mu^2}\right)$$
$$+\alpha_1\alpha_4\alpha_7 \left(\frac{-t}{\mu^2}\right).$$

Gaussian integrals

For a real symmetric positive definite $(n \times n)$-matrix A we have

$$\int\limits_{-\infty}^{\infty} dy_1 \ldots dy_n \, \exp\left(-\vec{y}^{\,T} A \vec{y} + 2\vec{w}^{\,T}\vec{y} + c\right) = \pi^{n/2} \left(\det A\right)^{-\frac{1}{2}} \exp\left(\vec{w}^{\,T} A^{-1} \vec{w} + c\right)$$

Schwinger parameter representation

Schwinger parameter representation:

$$I = \frac{e^{l\varepsilon\gamma_E}}{\prod\limits_{j=1}^{n_{\mathrm{int}}}\Gamma(\nu_j)} \int\limits_{\alpha_j \geq 0} d^{n_{\mathrm{int}}}\alpha \left(\prod_{j=1}^{n_{\mathrm{int}}} \alpha_j^{\nu_j - 1}\right) [\mathcal{U}(\alpha)]^{-\frac{D}{2}} \exp\left(-\frac{\mathcal{F}(\alpha)}{\mathcal{U}(\alpha)}\right)$$

- We went from a $(l \cdot D)$-fold momentum integration to a n_{int}-fold Schwinger parameter integration.
- The number of space-time dimensions D enters only the exponent of $\mathcal{U}^{-D/2}$ (and the prefactor).

Subsection 3

The Feynman parameter representation

From Schwinger to Feynman

The sum of the Schwinger parameters is non-negative:

$$\sum_{j=1}^{n} \alpha_j \quad \geq \quad 0.$$

We insert a 1 in the form of

$$1 \quad = \quad \int\limits_{-\infty}^{\infty} dt \, \delta\left(t - \sum_{j=1}^{n} \alpha_j\right) \; = \; \int\limits_{0}^{\infty} dt \, \delta\left(t - \sum_{j=1}^{n} \alpha_j\right),$$

where in the last step we used the fact that the sum of the Schwinger parameters is non-negative. $\delta(x)$ denotes Dirac's delta distribution.
Changing variables according to $a_j = \alpha_j / t$ gives us the identity

$$\int\limits_{\alpha_j \geq 0} d^n\alpha \, f(\alpha_1, \dots, \alpha_n) \quad = \quad \int\limits_{a_j \geq 0} d^n a \, \delta\left(1 - \sum_{j=1}^{n} a_j\right) \int\limits_{0}^{\infty} dt \; t^{n-1} \; f(ta_1, \dots, ta_n).$$

From Schwinger to Feynman

Apply the identity to the Schwinger parameter representation and use the fact that $\mathcal{U}$ and $\mathcal{F}$ are homogeneous of degree l and $(l+1)$, respectively:

$$
\begin{aligned}
I &= \frac{e^{l\varepsilon\gamma_E}}{\prod\limits_{j=1}^{n_{\mathrm{int}}} \Gamma(\nu_j)} \int\limits_{a_j \geq 0} d^{n_{\mathrm{int}}} a \, \delta\left(1 - \sum_{j=1}^{n_{\mathrm{int}}} a_j\right) \left(\prod_{j=1}^{n_{\mathrm{int}}} a_j^{\nu_j - 1}\right) [\mathcal{U}(a)]^{-\frac{D}{2}} \\
&\times \int\limits_0^\infty dt \, t^{\nu - \frac{lD}{2} - 1} \exp\left(-\frac{\mathcal{F}(a)}{\mathcal{U}(a)} t\right) \\
&= \frac{e^{l\varepsilon\gamma_E}}{\prod\limits_{j=1}^{n_{\mathrm{int}}} \Gamma(\nu_j)} \int\limits_{a_j \geq 0} d^{n_{\mathrm{int}}} a \, \delta\left(1 - \sum_{j=1}^{n_{\mathrm{int}}} a_j\right) \left(\prod_{j=1}^{n_{\mathrm{int}}} a_j^{\nu_j - 1}\right) \frac{[\mathcal{U}(a)]^{\nu - \frac{(l+1)D}{2}}}{[\mathcal{F}(a)]^{\nu - \frac{lD}{2}}} \int\limits_0^\infty dt \, t^{\nu - \frac{lD}{2} - 1} e^{-t}.
\end{aligned}
$$

In the step towards the last line we substituted $t \to t\,\mathcal{U}(a)/\mathcal{F}(a)$. The final integral over t gives $\Gamma(\nu - lD/2)$.

Feynman parameter representation

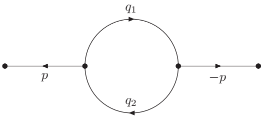
Feynman parameter representation:

$$I = \frac{e^{l\varepsilon\gamma_\mathrm{E}}\Gamma\left(\nu - \frac{lD}{2}\right)}{\prod\limits_{j=1}^{n_\mathrm{int}}\Gamma(\nu_j)} \int\limits_{a_j \geq 0} d^{n_\mathrm{int}}a\, \delta\left(1 - \sum_{j=1}^{n_\mathrm{int}} a_j\right) \left(\prod_{j=1}^{n_\mathrm{int}} a_j^{\nu_j - 1}\right) \frac{[\mathcal{U}(a)]^{\nu - \frac{(l+1)D}{2}}}{[\mathcal{F}(a)]^{\nu - \frac{lD}{2}}}$$

The polynomials $\mathcal{U}$ and $\mathcal{F}$ are as before, with α_j substituted by a_j.
The variable a_j is called a **Feynman parameter**.

Example

Consider the one-loop bubble diagram with vanishing internal masses.
The graph polynomials are



$$\mathcal{U} = a_1 + a_2, \qquad \mathcal{F} = a_1 a_2 \left(\frac{-p^2}{\mu^2} \right).$$

$$I_{\nu_1 \nu_2} = \frac{e^{\varepsilon \gamma_E} \Gamma\left(\nu - \frac{D}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int d^2a\, \delta(1 - a_1 - a_2)\, a_1^{\nu_1 - 1} a_2^{\nu_2 - 1} \frac{\mathcal{U}^{\nu - D}}{\mathcal{F}^{\nu - \frac{D}{2}}}.$$

Example

$$
\begin{aligned}
I_{\nu_1\nu_2} &= \frac{e^{\varepsilon\gamma_E}\Gamma\left(\nu-\frac{D}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int d^2a\,\delta(1-a_1-a_2)\,a_1^{\nu_1-1}a_2^{\nu_2-1}\frac{\mathcal{U}^{\nu-D}}{\mathcal{F}^{\nu-\frac{D}{2}}} \\
&= \frac{e^{\varepsilon\gamma_E}\Gamma\left(\nu-\frac{D}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \left(\frac{-p^2}{\mu^2}\right)^{\frac{D}{2}-\nu} \int\limits_0^1 da\, a^{\frac{D}{2}-\nu_2-1}(1-a)^{\frac{D}{2}-\nu_1-1} \\
&= e^{\varepsilon\gamma_E}\frac{\Gamma\left(\nu-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-\nu_1\right)\Gamma\left(\frac{D}{2}-\nu_2\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(D-\nu)}\left(\frac{-p^2}{\mu^2}\right)^{\frac{D}{2}-\nu}.
\end{aligned}
$$

Specialising to $\nu_1 = \nu_2 = 1$, $D = 4-2\varepsilon$ and setting $L = \ln(-p^2/\mu^2)$:

$$
I_{11} = \frac{1}{\varepsilon} + 2 - L + \mathcal{O}(\varepsilon).
$$

Quiz

The integral

$$\int\limits_{a_j \geq 0} d^3a \, \delta(1 - a_1 - a_2 - a_3) \; f(a_1, a_2, a_3)$$

can be re-written as

$$\text{(A)} \quad \int\limits_0^1 da_1 \int\limits_0^1 da_2 \; f(a_1, a_2, 1 - a_1 - a_2)$$

$$\text{(B)} \quad \int\limits_0^1 da_1 \int\limits_0^{a_1} da_2 \; f(a_1, a_2, 1 - a_1 - a_2)$$

$$\text{(C)} \quad \int\limits_0^1 da_1 \int\limits_0^{1-a_1} da_2 \; f(a_1, a_2, 1 - a_1 - a_2)$$

$$\text{(D)} \quad \int\limits_0^\infty da_1 \int\limits_0^\infty da_2 \; f(a_1, a_2, 1 - a_1 - a_2)$$

The Cheng-Wu theorem

Let S be a non-empty subset of $\{1, \dots, n_{\mathrm{int}}\}$. We may replace

$$\delta\left(1 - \sum_{j=1}^{n_{\mathrm{int}}} a_j\right) \quad \textit{by} \quad \delta\left(1 - \sum_{j \in S} a_j\right).$$

The Feynman integral is then given by

$$I = \frac{e^{l\varepsilon\gamma_{\mathrm{E}}}\Gamma\left(\nu - \frac{lD}{2}\right)}{\prod\limits_{j=1}^{n_{\mathrm{int}}}\Gamma(\nu_j)} \int\limits_{a_j \geq 0} d^{n_{\mathrm{int}}}a\, \delta\left(1 - \sum_{j \in S} a_j\right) \left(\prod_{j=1}^{n_{\mathrm{int}}} a_j^{\nu_j - 1}\right) \frac{\left[\mathcal{U}(a)\right]^{\nu - \frac{(l+1)D}{2}}}{\left[\mathcal{F}(a)\right]^{\nu - \frac{lD}{2}}}.$$

In particular one may choose $S = \{j_0\}$, which sets a_{j_0} to one. The integration is then over all other Feynman parameters from zero to infinity.

Projective space

Let $\mathbb{F}$ be a field, typically $\mathbb{R}$ or $\mathbb{C}$.
The projective space $\mathbb{FP}^n$ is the set of lines through the origin in $\mathbb{F}^{n+1}$.
Equivalently, it is the set of points in $\mathbb{F}^{n+1}\backslash\{0\}$ modulo the equivalence relation

$$(x_0, x_1, ..., x_n) \sim (y_0, y_1, ..., y_n) \quad \Leftrightarrow \quad \exists\, \lambda \neq 0 : (x_0, x_1, ..., x_n) = (\lambda y_0, \lambda y_1, ..., \lambda y_n).$$

Points in $\mathbb{FP}^n$ will be denoted by

$$[z_0 : z_1 : ... : z_n].$$

These coordinates are called **homogeneous coordinates**.

Projective space

The **positive real projective space** $\mathbb{RP}^n_{>0}$ is the set of all points of $\mathbb{RP}^n$, which can be represented by

$$\left[x_0 : x_1 : ... : x_n\right] \quad \text{with} \quad x_i \; > \; 0, \qquad 0 \le j \le n.$$

We have $[1 : 2 : 3] \in \mathbb{RP}^2_{>0}$ and $[(-4) : (-5) : (-6)] \in \mathbb{RP}^2_{>0}$, but $[7 : (-8) : 9] \notin \mathbb{RP}^2_{>0}$.

The **non-negative real projective space** $\mathbb{RP}^n_{\ge 0}$ is the set of all points of $\mathbb{RP}^n$, which can be represented by

$$\left[x_0 : x_1 : ... : x_n\right] \quad \text{with} \quad x_i \; \ge \; 0, \qquad 0 \le j \le n.$$

Projective space

Define a **function**

$$f(a) = \frac{e^{l\varepsilon\gamma_{\mathrm{E}}}\Gamma\left(\nu - \frac{lD}{2}\right)}{\prod\limits_{j=1}^{n_{\mathrm{int}}}\Gamma(\nu_j)} \left(\prod_{j=1}^{n_{\mathrm{int}}} a_j^{\nu_j-1}\right) \frac{\left[\mathcal{U}(a)\right]^{\nu-\frac{(l+1)D}{2}}}{\left[\mathcal{F}(a)\right]^{\nu-\frac{lD}{2}}},$$

and a **differential $(n_{\mathrm{int}}-1)$-form**

$$\omega = \sum_{j=1}^{n_{\mathrm{int}}} (-1)^{n_{\mathrm{int}}-j}\, a_j\, da_1 \wedge \dots \wedge \widehat{da_j} \wedge \dots \wedge da_{n_{\mathrm{int}}},$$

where the hat indicates that the corresponding term is omitted.

Projective space

Projective Feynman parameter integral representation*:*

$$I = \int\limits_{\mathbb{RP}^{n_{\mathrm{int}}-1}_{\geq 0}} f\omega,$$

where $\mathbb{RP}^{n_{\mathrm{int}}-1}_{\geq 0}$ *denotes the non-negative real projective space of dimension* $(n_{\mathrm{int}} - 1)$*.*

In particular we may integrate over any hyper-surface covering the solid angle $a_j \geq 0$.

Subsection 4

The Lee-Pomeransky representation

Lee-Pomeransky representation

Lee-Pomeransky representation*:*

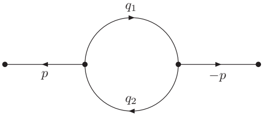
$$I = \frac{e^{l\varepsilon\gamma_E}\,\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{(l+1))D}{2}-\nu\right)\prod\limits_{j=1}^{n_{\mathrm{int}}}\Gamma(\nu_j)} \int\limits_{u_j\geq 0} d^{n_{\mathrm{int}}}u \left(\prod_{j=1}^{n_{\mathrm{int}}} u_j^{\nu_j-1}\right) \left[\mathcal{G}(u)\right]^{-\frac{D}{2}},$$

$$\textit{with} \quad \mathcal{G}(u) = \mathcal{U}(u) + \mathcal{F}(u).$$

- Only one polynomial $\mathcal{G}$ enters, given by the sum of the two graph polynomials $\mathcal{G} = \mathcal{U} + \mathcal{F}$.
- Proof: Insert 1 in the form of an integral over a Dirac delta distribution.

Example

Consider the one-loop bubble diagram with vanishing internal masses.



The Lee-Pomeransky polynomial is

$$\mathcal{G} = u_1 + u_2 + u_1 u_2 x,$$

where we set $x = -p^2/\mu^2$.

The Lee-Pomeransky representation is given by

$$I_{\nu_1\nu_2} = \frac{e^{\varepsilon\gamma_E}\Gamma\left(\frac{D}{2}\right)}{\Gamma(D-\nu)\Gamma(\nu_1)\Gamma(\nu_2)} \int\limits_0^\infty du_1 \int\limits_0^\infty du_2 \, u_1^{\nu_1-1} u_2^{\nu_2-1} \left[u_1 + u_2 + u_1 u_2 x\right]^{-\frac{D}{2}}.$$

Subsection 5

The Baikov representation

The Baikov representation

Let $p_1, p_2, ..., p_{n_\text{ext}}$ denote the external momenta and denote by

$$e = \dim \langle p_1, p_2, ..., p_{n_\text{ext}} \rangle$$

the dimension of the span of the external momenta.
For generic external momenta and $D \geq n_\text{ext} - 1$ we have $e = n_\text{ext} - 1$.
Lorentz invariants involving the loop momenta are of the form

$$\begin{aligned}
-k_i^2, &\quad 1 \leq i \leq l, \\
-k_i \cdot k_j, &\quad 1 \leq i < j \leq l, \\
-k_i \cdot p_j, &\quad 1 \leq i \leq l, \quad 1 \leq j \leq e.
\end{aligned}$$

In total we have

$$N_V = \frac{1}{2} l (l+1) + el$$

linear independent scalar products involving the loop momenta. Set

$$\sigma = (\sigma_1, ..., \sigma_{N_V}) = (-k_1 \cdot k_1, -k_1 \cdot k_2, ..., -k_l \cdot k_l, -k_1 \cdot p_1, ..., -k_l \cdot p_e).$$

The Baikov representation

A Feynman graph G has a Baikov representation if

1. $$N_V = n_{\mathrm{int}}$$

2. Any internal inverse propagator can be expressed as a linear combination of the linear independent scalar products involving the loop momenta and terms independent of the loop momenta.

The second condition says, that there is an invertible $N_V \times N_V$-dimensional matrix C and a loop-momentum independent N_V-dimensional vector f such that

$$-q_s^2 + m_s^2 = C_{st}\sigma_t + f_s$$

for all $1 \leq s \leq n_{\mathrm{int}}$.

Gram determinants

Define

$$\det G(q_1, ..., q_n) = \det(-q_i \cdot q_j) = \det(Q_i \cdot Q_j),$$

e.g.

$$\det G(q_1, q_2) = \begin{vmatrix} -q_1^2 & -q_1 \cdot q_2 \\ -q_1 \cdot q_2 & -q_2^2 \end{vmatrix} = q_1^2 q_2^2 - (q_1 \cdot q_2)^2.$$

We change the integration variables to the **Baikov variables** z_j:

$$z_j = -q_j^2 + m_j^2.$$

The determinant $\det G(k_1, ..., k_l, p_1, ..., p_e)$ expressed in the variables z_s's is called the **Baikov polynomial**:

$$\mathcal{B}(z_1, ..., z_{N_V}) = \det G(k_1, ..., k_l, p_1, ..., p_e).$$

The Baikov representation

Baikov representation:

$$I = \frac{e^{l\varepsilon\gamma_E} \left(\mu^2\right)^{\nu - \frac{lD}{2}} \left[\det G(p_1, ..., p_e)\right]^{\frac{-D+e+1}{2}}}{\pi^{\frac{1}{2}(N_V - l)} (\det C) \prod\limits_{j=1}^{l} \Gamma\left(\frac{D-e+1-j}{2}\right)} \int\limits_{\mathcal{C}} d^{N_V} z \, \left[\mathcal{B}(z)\right]^{\frac{D-l-e-1}{2}} \prod_{s=1}^{N_V} z_s^{-\nu_s}.$$

The domain of integration $\mathcal{C}$ is given by

$$\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2 \cap \dots \cap \mathcal{C}_l$$

with

$$\mathcal{C}_j = \left\{ \frac{\det G(k_j, k_{j+1}, ..., k_l, p_1, ..., p_e)}{\det G(k_{j+1}, ..., k_l, p_1, ..., p_e)} \geq 0 \right\}.$$

Comments

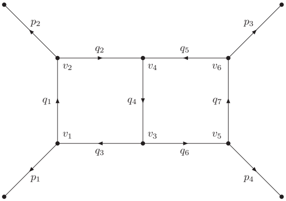
- Sketch of the proof:
  - Decompose loop momenta $k = k_{\|} + k_{\perp}$ into a parallel space $k_{\|} \in \langle p_1, p_2, ..., p_{n_{\text{ext}}} \rangle$ and a component living in the complement, called the orthogonal space.
  - Integrate out the orthogonal space.
  - Change variables in the parallel space to the Baikov variables.
  - The Gram determinants enter through the Jacobian.
- The Baikov representation is very convenient for computing **cuts** of Feynman integrals. This corresponds to taking residues in the Baikov representation.

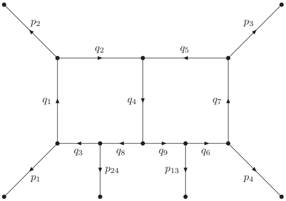
Comments

Suppose that the number of internal propagators n_{int} is smaller than the number N_V of linear independent scalar products involving the loop momenta:

$$n_{\mathrm{int}} \quad < \quad N_V.$$

This does not occur at one-loop, but it frequently occurs beyond one-loop. In this situation consider for the Baikov representation a **larger graph** $\tilde{G}$, with the property that the original graph G is as a subgraph of $\tilde{G}$.





Irreducible scalar products

The two-loop double box graph has seven propagators ($n_{\mathrm{int}} = 7$), but nine linear independent scalar products involving the loop momenta:

$$-k_1^2, \; -k_2^2, \; -k_1 \cdot k_2, \; -k_1 \cdot p_1, \; -k_1 \cdot p_2, \; -k_1 \cdot p_3, \; -k_2 \cdot p_1, \; -k_2 \cdot p_2, \; -k_2 \cdot p_3.$$

We may express seven scalar products

$$-k_1^2, \; -k_2^2, \; -k_1 \cdot k_2, \; -k_1 \cdot p_1, \; -k_1 \cdot p_2, \; -k_2 \cdot (p_1 + p_2), \; -k_2 \cdot p_3$$

in terms of inverse propagators, but not

$$-k_1 \cdot p_3, \qquad -k_2 \cdot p_1.$$

These scalar products are called **irreducible scalar products**.

Example

We consider the one-loop tapole integral

$$T_\nu(D,x) = e^{\varepsilon\gamma_E}\left(\mu^2\right)^{\nu-\frac{D}{2}} \int \frac{d^Dk}{i\pi^{\frac{D}{2}}} \frac{1}{\left(-k^2+m^2\right)^\nu}, \quad x = \frac{m^2}{\mu^2}.$$

This integral does not depend on any external momenta, therefore

$$e = 0 \quad \text{and} \quad N_V = 1.$$

There is one Baikov variable $z_1 = -k^2+m^2$. The Baikov polynomial is given by

$$\mathcal{B}(z_1) = \det G(k) = -k^2 = z_1 - m^2.$$

Baikov representation:

$$T_\nu(D,x) = \frac{e^{\gamma_E\varepsilon}\left(\mu^2\right)^{\nu-\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int\limits_{\mathcal{C}} dz_1 \left[\mathcal{B}(z_1)\right]^{\frac{D}{2}-1} \frac{1}{z_1^\nu},$$

The requirement $\det G(k) \geq 0$ defines the integration region $\mathcal{C} = [m^2, \infty[$.

Subsection 6

The Mellin-Barnes representation

Motivation

The Feynman parameter representation depends on two graph polynomials $\mathcal{U}$ and $\mathcal{F}$.

Assume for the moment that the two graph polynomials $\mathcal{U}$ and $\mathcal{F}$ are absent.

In this case we have an integral of the form

$$\int\limits_{a_j \geq 0} d^n a \, \delta\left(1 - \sum_{j=1}^{n} a_j\right) \left(\prod_{j=1}^{n} a_j^{\nu_j - 1}\right) = \frac{\prod\limits_{j=1}^{n} \Gamma(\nu_j)}{\Gamma(\nu_1 + ... + \nu_n)}.$$

The Mellin-Barnes transformation allows us to reduce the Feynman parameter integration to this case.

The Mellin-Barnes formula

The basic formula:

$$(A+B)^{-c} \quad = \quad \frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} d\sigma \, \frac{\Gamma(-\sigma)\Gamma(\sigma+c)}{\Gamma(c)} \, A^{\sigma} B^{-\sigma-c}.$$

Iterated:

$$(A_1+A_2+...+A_n)^{-c} = \frac{1}{\Gamma(c)} \frac{1}{(2\pi i)^{n-1}} \int\limits_{-i\infty}^{i\infty} d\sigma_1 ... \int\limits_{-i\infty}^{i\infty} d\sigma_{n-1}$$

$$\times \Gamma(-\sigma_1)...\Gamma(-\sigma_{n-1})\Gamma(\sigma_1+...+\sigma_{n-1}+c) \, A_1^{\sigma_1}...A_{n-1}^{\sigma_{n-1}} A_n^{-\sigma_1-...-\sigma_{n-1}-c}.$$

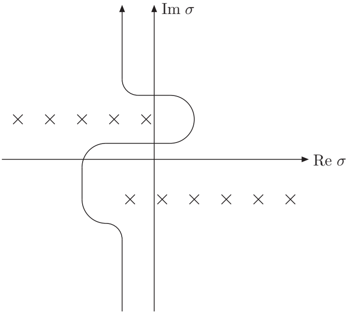
Each contour is such that the poles of $\Gamma(-\sigma)$ are to the right and the poles of $\Gamma(\sigma+c)$ are to the left.

This transformation can be used to convert the sum of monomials of the polynomials $\mathcal{U}$ and $\mathcal{F}$ into a product.

The integration contour

A typical single contour integral is of the form

$$I = \frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} d\sigma \, \frac{\Gamma(\sigma+a_1)...\Gamma(\sigma+a_m)}{\Gamma(\sigma+c_1)...\Gamma(\sigma+c_p)} \frac{\Gamma(-\sigma+b_1)...\Gamma(-\sigma+b_n)}{\Gamma(-\sigma+d_1)...\Gamma(-\sigma+d_q)} \, x^{-\sigma}.$$



Evaluating Mellin-Barnes integrals

A Mellin-Barnes integral is most conveniently evaluated with the help of the **residuum theorem** by closing the contour to the left or to the right.

To sum up all residues which lie inside the contour it is useful to know the residues of the Gamma function:

$$\text{res}\ \left(\Gamma(\sigma+a),\sigma=-a-n\right)=\frac{(-1)^n}{n!},\qquad \text{res}\ \left(\Gamma(-\sigma+a),\sigma=a+n\right)=-\frac{(-1)^n}{n!}.$$

In the case where we close the contour to the right, there is an extra minus sign from the negative winding number.