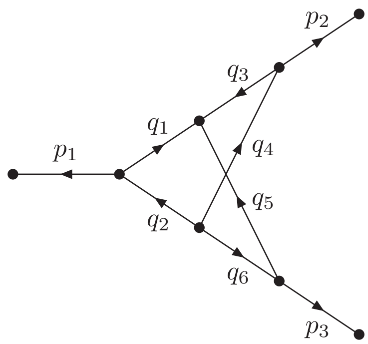
# Exercises for the course "Feynman integrals"
## Sheet 2

## Exercise 3

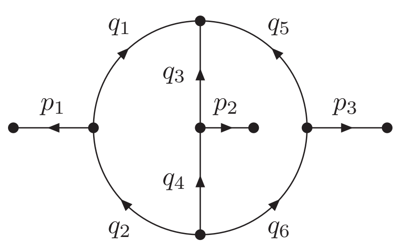
Determine the graph polynomials $\mathcal{U}$ and $\mathcal{F}$ for the graph




for the case where all internal masses are zero (but $p_1^2 \neq 0$, $p_2^2 \neq 0$, $p_3^2 \neq 0$).

## **Solution**:

Let us first note that the graph may equally well be drawn as




As independent loop momenta we take

$$k_1 = q_1, \qquad k_2 = q_5.$$

Then

$$q_2 = k_1 + p_1, \qquad q_3 = -k_1 - k_2, \qquad q_4 = -k_1 - k_2 + p_2, \qquad q_6 = k_2 + p_3,$$

and $p_3 = -p_1 - p_2$. We work out

$$\sum_{j=1}^{6} \alpha_j \left(-q_j^2\right) = -(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\,k_1^2 - 2\,(\alpha_3+\alpha_4)\,k_1 \cdot k_2 - (\alpha_3+\alpha_4+\alpha_5+\alpha_6)\,k_2^2$$
$$+2\,(\alpha_4 p_2 - \alpha_2 p_1) \cdot k_1 + 2\,(\alpha_4 p_2 - \alpha_6 p_3) \cdot k_2 - \alpha_2 p_1^2 - \alpha_4 p_2^2 - \alpha_6 p_3^2.$$

We find

$$
\begin{aligned}
M &= \begin{pmatrix} \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \alpha_3+\alpha_4 \\ \alpha_3+\alpha_4 & \alpha_3+\alpha_4+\alpha_5+\alpha_6 \end{pmatrix}, \\
v &= \begin{pmatrix} \alpha_4 p_2 - \alpha_2 p_1 \\ \alpha_4 p_2 - \alpha_6 p_3 \end{pmatrix}, \\
J &= \alpha_2 \left(-p_1\right)^2 + \alpha_4 \left(-p_2\right)^2 + \alpha_6 \left(-p_3\right)^2 .
\end{aligned}
$$

From momentum conservation we have $(p_1+p_2)^2 = p_3^2$ and hence

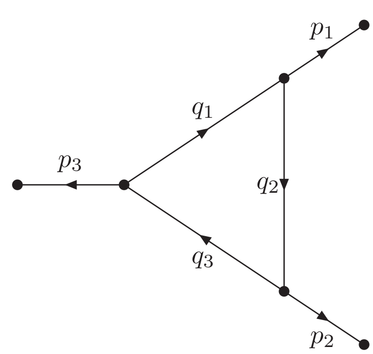
$$
\begin{aligned}
2p_1 \cdot p_2 &= p_3^2 - p_1^2 - p_2^2, \\
2p_2 \cdot p_3 &= p_1^2 - p_2^2 - p_3^2, \\
2p_3 \cdot p_1 &= p_2^2 - p_3^2 - p_1^2.
\end{aligned}
$$

Plugging everything together we obtain

$$
\begin{aligned}
\mathcal{U} &= (\alpha_1+\alpha_2)(\alpha_3+\alpha_4) + (\alpha_1+\alpha_2)(\alpha_5+\alpha_6) + (\alpha_3+\alpha_4)(\alpha_5+\alpha_6), \\
\mathcal{F} &= \left[\alpha_1\alpha_4\alpha_6 + \alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_2\left(\alpha_3+\alpha_4+\alpha_5\alpha_6\right)\right]\left(\frac{-p_1^2}{\mu^2}\right) \\
&\quad + \left[\alpha_3\alpha_2\alpha_6 + \alpha_4\alpha_1\alpha_5 + \alpha_3\alpha_4\left(\alpha_1+\alpha_2+\alpha_5\alpha_6\right)\right]\left(\frac{-p_2^2}{\mu^2}\right) \\
&\quad + \left[\alpha_5\alpha_2\alpha_4 + \alpha_6\alpha_1\alpha_3 + \alpha_5\alpha_6\left(\alpha_1+\alpha_2+\alpha_3\alpha_4\right)\right]\left(\frac{-p_3^2}{\mu^2}\right).
\end{aligned}
$$

## Exercise 4

Calculate with the help of the Feynman parameter representation the one-loop triangle integral




$$
I_{\nu_1\nu_2\nu_3} = e^{\varepsilon\gamma_E}\left(\mu^2\right)^{\nu-\frac{D}{2}} \int \frac{d^Dk}{i\pi^{\frac{D}{2}}} \frac{1}{\left(-q_1^2\right)^{\nu_1}\left(-q_2^2\right)^{\nu_2}\left(-q_3^2\right)^{\nu_3}},
$$

for the case where all internal masses are zero ($m_1 = m_2 = m_3 = 0$) and for the kinematic configuration $p_1^2 = p_2^2 = 0$, $p_3^2 \neq 0$.

**Solution**:

The second graph polynomial is given by

$$\mathcal{F} = a_1 a_3 \left( \frac{-p_3^2}{\mu^2} \right)$$

and the Feynman parameter representation reads

$$\begin{aligned} I_{\nu_1\nu_2\nu_3} &= \frac{e^{\varepsilon\gamma_E}\Gamma\left(\nu-\frac{D}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int\limits_{a_j\geq 0} d^3a\, \delta(1-a_1-a_2-a_3)\, \frac{a_1^{\nu_1-1}a_2^{\nu_2-1}a_3^{\nu_3-1}}{\left[\mathcal{F}(a)\right]^{\nu-\frac{D}{2}}} \\ &= \frac{e^{\varepsilon\gamma_E}\Gamma\left(\nu-\frac{D}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \left(\frac{-p_3^2}{\mu^2}\right)^{\frac{D}{2}-\nu} \int\limits_{a_j\geq 0} d^3a\, \delta(1-a_1-a_2-a_3)\, a_1^{\frac{D}{2}-\nu_{23}-1} a_2^{\nu_2-1} a_3^{\frac{D}{2}-\nu_{12}-1}. \end{aligned}$$

The Feynman parameter integral is a generalisation of Euler's beta function: For $n \in \mathbb{N}$ we have

$$\int\limits_{a_j\geq 0} d^na\, \delta\left(1-\sum_{j=1}^{n} a_j\right) \left(\prod_{j=1}^{n} a_j^{\nu_j-1}\right) = \frac{\prod\limits_{j=1}^{n}\Gamma(\nu_j)}{\Gamma(\nu_1+...+\nu_n)}$$

and therefore

$$I_{\nu_1\nu_2\nu_3} = \frac{e^{\varepsilon\gamma_E}\Gamma\left(\nu-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-\nu_{12}\right)\Gamma\left(\frac{D}{2}-\nu_{23}\right)}{\Gamma(\nu_1)\Gamma(\nu_3)\Gamma(D-\nu)} \left(\frac{-p_3^2}{\mu^2}\right)^{\frac{D}{2}-\nu}.$$