Feynman Integrals

Lecture 3

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Section 1

Graph polynomials

Subsection 1

Definition as it is done in textbooks of physics

The textbook definition

Write

$$\sum_{j=1}^{n_{\text{int}}} a_j \left(-q_j^2 + m_j^2\right) = -\sum_{r=1}^{l} \sum_{s=1}^{l} k_r M_{rs} k_s + \sum_{r=1}^{l} 2 k_r \cdot v_r + J$$

M : $l \times l$ matrix with scalar entries

v : l-vector with D-dimensional momentum vectors as entries

J : scalar

$$\mathcal{U} = \det(M) \qquad \mathcal{F} = \det(M) \left(J + v^T M^{-1} v \right) / \mu^2$$

Subsection 2

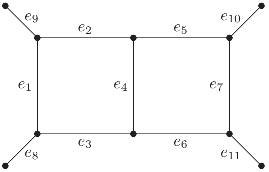
Definition through spanning trees and spanning forests

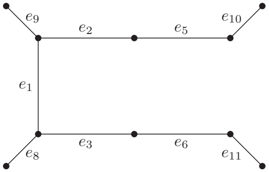
Spanning trees

Definition

A **spanning tree** for the graph G is a sub-graph T of G satisfying the following requirements:

- T contains all the vertices of G,
- the first Betti number of T is zero,
- T is connected.



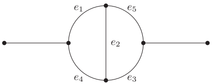


The first graph polynomial

Let G be a connected graph and $\mathcal{T}_1$ the set of its spanning trees.
The **first graph polynomial** *is given by*

$$\mathcal{U}(a) \;=\; \sum_{T \in \mathcal{T}_1} \prod_{e_i \notin T} a_i,$$

Example:



$$a_1 a_5 + a_3 a_4 + a_1 a_3 + a_4 a_5 + a_2 a_5 + a_1 a_2 + a_2 a_4 + a_2 a_3$$

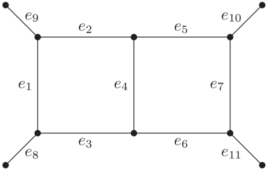
Spanning forests

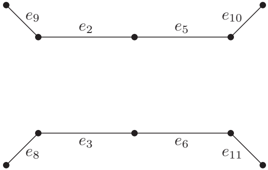
Definition

A **spanning forest** for the graph *G* with respect to an edge set *E* is a sub-graph *F* of *G* satisfying:

- *F* contains all the vertices of *G*,
- the first Betti number of *F* is zero.
- *F* contains all edges $\{e_1, \dots, e_n\} \backslash E$.

Thus we are only allowed to remove edges which belong to *E*.
Typical application: $E =$ set of all internal edges.





The second graph polynomial

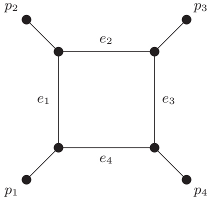
Let G be a connected graph and $\mathcal{T}_2$ the set of its spanning 2-forests with respect to the internal edges.
An element of $\mathcal{T}_2$ is denoted as (T_1, T_2).
Let further denote P_{T_i} the set of external momenta of G attached to T_i.
The **second graph polynomial** *is given by*

$$\mathcal{F}(a) = \mathcal{F}_0(a) + \mathcal{U}(a) \sum_{i=1}^{n_{\mathrm{int}}} a_i \frac{m_i^2}{\mu^2},$$

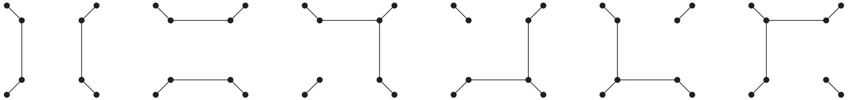
$$\mathcal{F}_0(a) = \sum_{(T_1,T_2)\in\mathcal{T}_2} \left(\prod_{e_i \notin (T_1,T_2)} a_i \right) \left(\sum_{p_j \in P_{T_1}} \sum_{p_k \in P_{T_2}} \frac{p_j \cdot p_k}{\mu^2} \right).$$

The sum is over all spanning 2-forests (with respect to the internal edges).

Example



$$m_1 = m_2 = m_3 = m_4 = 0$$
$$s = (p_1 + p_2)^2$$
$$t = (p_2 + p_3)^2$$



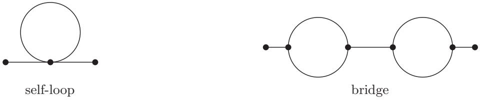
$$a_2 a_4 \frac{(-s)}{\mu^2} + a_1 a_3 \frac{(-t)}{\mu^2} + a_1 a_4 \frac{(-p_1^2)}{\mu^2} + a_1 a_2 \frac{(-p_2^2)}{\mu^2} + a_2 a_3 \frac{(-p_3^2)}{\mu^2} + a_3 a_4 \frac{(-p_4^2)}{\mu^2}$$

Subsection 3

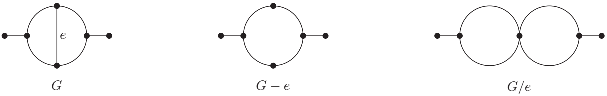
Definition through deletion and contraction properties

Deletion and contraction

Regular edge: Neither a self-loop nor a bridge



G/e	Graph obtained from G by contracting the regular edge e,
$G-e$	Graph obtained from G by deleting the regular edge e.



Deletion and contraction properties

Recursive definition of the Feynman graph polynomials for massless particles:
For any regular edge e_k we have

$$\begin{aligned} \mathcal{U}(G) &= \mathcal{U}(G/e_k) + a_k \mathcal{U}(G - e_k), \\ \mathcal{F}(G) &= \mathcal{F}(G/e_k) + a_k \mathcal{F}(G - e_k). \end{aligned}$$

Recursion terminates when all edges are either bridges or self-loops.
For a terminal form we have

$$\mathcal{U} = a_r ... a_n, \qquad \mathcal{F} = a_r ... a_n \sum_{j=1}^{r-1} a_j \left(\frac{-q_j^2}{\mu^2} \right),$$

where we labelled the edges, which are bridges from 1 to $(r-1)$, and the ones which are self-loops from r to n.

q_j is the momentum flowing throught the bridge j.

Section 2

Tensor reduction

Tensor integrals

$$I_{\nu_1 \dots \nu_{n_{\text{int}}}}\left(D, x_1, \dots, x_{N_B}\right)\left[k_{i_1}^{\mu_1} \dots k_{i_t}^{\mu_t}\right] = e^{l \varepsilon \gamma_{\text{E}}} \left(\mu^2\right)^{\nu - \frac{lD}{2}} \int \prod_{r=1}^{l} \frac{d^D k_r}{i \pi^{\frac{D}{2}}} \frac{k_{i_1}^{\mu_1} \dots k_{i_t}^{\mu_t}}{\prod\limits_{j=1}^{n_{\text{int}}} \left(-q_j^2 + m_j^2\right)^{\nu_j}}$$

Tensor integrals can be reduced to **scalar integrals**.

- For one-loop integrals we may use Passarino-Veltman reduction.
- We discuss here the general algorithm due to Tarasov.

Dimensional-shift operators and raising operators

Dimensional-shift operators:

$$\mathbf{D}^{\pm} I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B}) = I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D \pm 2, x_1, \dots, x_{N_B})$$

Raising operators:

$$\mathbf{j}^{+} I_{\nu_1 \dots \nu_j \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B}) = \nu_j \cdot I_{\nu_1 \dots (\nu_j + 1) \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B})$$

Note that we defined $\mathbf{j}^{+}$ such that it raises the index $\nu_j \to \nu_j + 1$ and multiplies the integral with a factor ν_j.

With this definition we have for example

$$\left(\mathbf{j}^{+}\right)^2 I_{\nu_1 \dots \nu_j \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B}) = \nu_j (\nu_j + 1) \cdot I_{\nu_1 \dots (\nu_j + 2) \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B}).$$

Dimensional-shift operators and raising operators

These operators act on the Schwinger representation

$$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D) = \frac{e^{l\varepsilon\gamma_{\mathrm{E}}}}{\prod\limits_{k=1}^{n_{\mathrm{int}}} \Gamma(\nu_k)} \int\limits_{\alpha_k \geq 0} d^{n_{\mathrm{int}}}\alpha \left(\prod_{k=1}^{n_{\mathrm{int}}} \alpha_k^{\nu_k - 1} \right) \frac{1}{\mathcal{U}^{\frac{D}{2}}} e^{-\frac{\mathcal{F}}{\mathcal{U}}}$$

as follows:

$$\mathbf{D}^+ I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D) = \frac{e^{l\varepsilon\gamma_{\mathrm{E}}}}{\prod\limits_{k=1}^{n_{\mathrm{int}}} \Gamma(\nu_k)} \int\limits_{\alpha_k \geq 0} d^{n_{\mathrm{int}}}\alpha \left(\prod_{k=1}^{n_{\mathrm{int}}} \alpha_k^{\nu_k - 1} \right) \frac{1}{\mathcal{U} \cdot \mathcal{U}^{\frac{D}{2}}} e^{-\frac{\mathcal{F}}{\mathcal{U}}},$$

$$\mathbf{j}^+ I_{\nu_1 \dots \nu_j \dots \nu_{n_{\mathrm{int}}}}(D) = \frac{e^{l\varepsilon\gamma_{\mathrm{E}}}}{\prod\limits_{k=1}^{n_{\mathrm{int}}} \Gamma(\nu_k)} \int\limits_{\alpha_k \geq 0} d^{n_{\mathrm{int}}}\alpha \left(\prod_{k=1}^{n_{\mathrm{int}}} \alpha_k^{\nu_k - 1} \right) \frac{\alpha_{\mathbf{j}}}{\mathcal{U}^{\frac{D}{2}}} e^{-\frac{\mathcal{F}}{\mathcal{U}}}.$$

Tensor reduction

Recall:

$$\sum_{j=1}^{n_{\text{int}}} \alpha_j(-q_j^2 + m_j^2) = -\sum_{r=1}^{l}\sum_{s=1}^{l} k_r M_{rs} k_s + \sum_{r=1}^{l} 2k_r \cdot v_r + J.$$

By a change of variables $k_r \to k_r'$ bring this quadric to the form

$$\sum_{j=1}^{n_{\text{int}}} \alpha_j(-q_j^2 + m_j^2) = -\sum_{r=1}^{l} \lambda_r k_r'^2 + J'.$$

This leads to integrals of the form

$$\int \frac{d^D k}{i\pi^{D/2}} k^{\mu_1} \dots k^{\mu_t} f(k^2).$$

Tensor reduction

Integrals with an **odd power** of the loop momentum in the numerator vanish by symmetry:

$$\int \frac{d^D k}{i\pi^{D/2}} k^{\mu_1} \dots k^{\mu_{2t-1}} f(k^2) \quad = \quad 0, \qquad t \in \mathbb{N}.$$

Integrals with an **even power** of the loop momentum must be proportional to a symmetric tensor build from the metric tensor due to Lorentz symmetry. For the simplest cases we have

$$\int \frac{d^D k}{i\pi^{D/2}} k^{\mu} k^{\nu} f(k^2) \quad = \quad -\frac{1}{D} g^{\mu\nu} \int \frac{d^D k}{i\pi^{D/2}} (-k^2) f(k^2),$$

$$\int \frac{d^D k}{i\pi^{D/2}} k^{\mu} k^{\nu} k^{\rho} k^{\sigma} f(k^2) \quad = \quad \frac{1}{D(D+2)} \left(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} \right) \int \frac{d^D k}{i\pi^{D/2}} (-k^2)^2 f(k^2).$$

Tensor reduction

The change of variables $k_r \to k_r'$ and the integration

$$\int \frac{d^D k}{i\pi^{D/2}} \left(-k^2\right)^a e^{\lambda k^2} \quad = \quad \frac{\Gamma\left(\frac{D}{2}+a\right)}{\Gamma\left(\frac{D}{2}\right)} \frac{1}{\lambda^{\frac{D}{2}+a}}$$

may

- introduce additional powers of the Schwinger parameters in the numerator and
- additional powers of the first graph polynomial $\mathcal{U}$ in the denominator.

We may write these integrals as scalar integrals with **raised powers** of the propagators and **shifted space-time dimensions**.

Section 3

Integration by parts

Integration by parts

Integration-by-parts identities are based on the fact that within dimensional regularisation the **integral of a total derivative vanishes**

$$\int \frac{d^D k}{i\pi^{\frac{D}{2}}} \; \frac{\partial}{\partial k^\mu} \left[q^\mu \cdot f(k) \right] \;=\; 0,$$

i.e. there are no boundary terms.

Integration by parts

Integration-by-parts identities*:*

Within dimensional regularisation we have for any loop momentum k_i and any vector $q_{\mathrm{IBP}} \in \{p_1, ..., p_{N_{\mathrm{ext}}}, k_1, ..., k_l\}$

$$e^{l\varepsilon\gamma_{\mathrm{E}}} \left(\mu^2\right)^{\nu - \frac{lD}{2}} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial \mathbf{k_i}^{\mu}} q^{\mu}_{\mathrm{IBP}} \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}} = 0.$$

Working out the derivatives leads to **relations among integrals** *with different sets of indices $(\nu_1, \dots, \nu_{n_{\mathrm{int}}})$.*

Example

Consider the one-loop two-point function with an equal internal mass with $q_1 = k - p$, $q_2 = k$, $\mu^2 = m^2$ and $x = -p^2/m^2$.

$$I_{\nu_1\nu_2}(D,x) = e^{\varepsilon\gamma_E} \left(m^2\right)^{\nu_{12}-\frac{D}{2}} \int \frac{d^Dk}{i\pi^{\frac{D}{2}}} \frac{1}{\left(-q_1^2+m^2\right)^{\nu_1}\left(-q_2^2+m^2\right)^{\nu_2}}.$$

For $q_{\mathrm{IBP}} = p$ we obtain

$$\begin{aligned} 0 &= e^{\varepsilon\gamma_E} \left(m^2\right)^{\nu_{12}-\frac{D}{2}} p^\mu \int \frac{d^Dk}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k^\mu} \frac{1}{\left(-q_1^2+m^2\right)^{\nu_1}\left(-q_2^2+m^2\right)^{\nu_2}} \\ &= e^{\varepsilon\gamma_E} \left(m^2\right)^{\nu_{12}-\frac{D}{2}} \int \frac{d^Dk}{i\pi^{\frac{D}{2}}} \left[\frac{\nu_1\left(q_2^2-q_1^2-p^2\right)}{\left(-q_1^2+m^2\right)^{\nu_1+1}\left(-q_2^2+m^2\right)^{\nu_2}} \right. \\ &\qquad \left. + \frac{\nu_2\left(q_2^2-q_1^2+p^2\right)}{\left(-q_1^2+m^2\right)^{\nu_1}\left(-q_2^2+m^2\right)^{\nu_2+1}} \right] \\ &= \nu_1 \left[\mathbf{I}_{\nu_1\nu_2} - \mathbf{I}_{(\nu_1+1)(\nu_2-1)} + \mathbf{x}\mathbf{I}_{(\nu_1+1)\nu_2} \right] + \nu_2 \left[\mathbf{I}_{(\nu_1-1)(\nu_2+1)} - \mathbf{I}_{\nu_1\nu_2} - \mathbf{x}\mathbf{I}_{\nu_1(\nu_2+1)} \right]. \end{aligned}$$

Example

We have two possible choices $q_{\mathrm{IBP}} \in \{p, k\}$.

This gives us two equations, which we can arrange as

$$\begin{aligned} & \nu_1 x (4+x) I_{(\nu_1+1)\nu_2} = \\ & \qquad [2(-\nu_1+\nu_2)+(\nu_1+2\nu_2-D)x] I_{\nu_1\nu_2} + \nu_1 (2+x) I_{(\nu_1+1)(\nu_2-1)} - 2\nu_2 I_{(\nu_1-1)(\nu_2+1)}, \\ & \nu_2 x (4+x) I_{\nu_1(\nu_2+1)} = \\ & \qquad [2(\nu_1-\nu_2)+(2\nu_1+\nu_2-D)x] I_{\nu_1\nu_2} - 2\nu_1 I_{(\nu_1+1)(\nu_2-1)} + \nu_2 (2+x) I_{(\nu_1-1)(\nu_2+1)}. \end{aligned}$$

For $\nu_1 > 0$ and $\nu_2 > 0$ we may use either the first or the second equation to reduce the sum $\nu_1 + \nu_2$:
In both equations, the sum of the indices equals $\nu_1 + \nu_2 + 1$ on the left-hand side, while on the right-hand side the sum of the indices equals for all terms $\nu_1 + \nu_2$.

Example

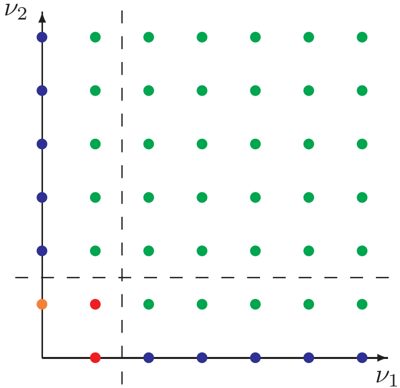
If either $\nu_1 = 0$ (and $\nu_2 > 0$) or $\nu_2 = 0$ (and $\nu_1 > 0$) we have a simpler integral: The integral reduces to a tadpole integral. As the two internal masses are equal, we have

$$I_{\nu_1 0} = I_{0\nu_2}.$$

For a tadpole integral we repeat the exercise of deriving integration-by-parts identities and we find

$$\nu_2 I_{0(\nu_2+1)} = \left(\nu_2 - \frac{D}{2}\right) I_{0\nu_2}.$$

Example



Integration-by-parts reduction for the one-loop two-point function:
We may reduce any integral $I_{\nu_1 \nu_2}$ to a linear combination of I_{11} and I_{10}.

Pinching of propagators

We consider integrals

$$I_{\nu_1 \dots \nu_{n_{\text{int}}}}, \qquad \nu_j \in \mathbb{Z}.$$

We call integrals, where all indices satisfy $\nu_j > 0$, integrals of the **top topology** (or of the **top sector**).
Integrals, where one or more integrals satisfy $\nu_j < 1$, belong to a sub-topology (or belong to a sub-sector).

The Laporta algorithm

For a given set of indices $(\nu_1, \dots, \nu_{n_{\mathrm{int}}})$ we define

$$N_{\mathrm{prop}} = \sum_{j=1}^{n_{\mathrm{int}}} \Theta(\nu_j > 0), \qquad N_{\mathrm{id}} = \sum_{j=1}^{n_{\mathrm{int}}} 2^{j-1} \Theta(\nu_j > 0),$$

$$r = \sum_{j=1}^{n_{\mathrm{int}}} \nu_j \Theta(\nu_j > 0), \qquad s = \sum_{j=1}^{n_{\mathrm{int}}} |\nu_j| \Theta(\nu_j < 0).$$

Order criteria: Lexicographical order of tuples

ISP-basis : $\left(N_{\mathrm{prop}}, N_{\mathrm{id}}, r, s, \dots\right)$

dot-basis : $\left(N_{\mathrm{prop}}, N_{\mathrm{id}}, s, r, \dots\right)$

Quiz

Consider the equation

$$I_{11103} + I_{111(-1)0} + I_{23500} + I_{11101} \quad = \quad 0.$$

With the order criteria $(N_{\mathrm{prop}}, N_{\mathrm{id}}, s, r)$ one eliminates

(A) I_{11103}

(B) $I_{111(-1)0}$

(C) I_{23500}

(D) I_{11101}

Master integrals

Using

- integration-by-parts identities
- symmetries

we may express most of the integrals in terms of a few remaining integrals.
The remaining integrals are called **master integrals**.
We denote the indices of the master integrals by

$$\begin{aligned}
\boldsymbol{\nu}_1 &= (\nu_{11}, \dots, \nu_{1 n_{\mathrm{int}}}), \\
\boldsymbol{\nu}_2 &= (\nu_{21}, \dots, \nu_{2 n_{\mathrm{int}}}), \\
& \dots \\
\boldsymbol{\nu}_{N_{\mathrm{master}}} &= (\nu_{N_{\mathrm{master}} 1}, \dots, \nu_{N_{\mathrm{master}} n_{\mathrm{int}}}).
\end{aligned}$$

We define a N_{master}-dimensional vector $\vec{I}$ by

$$\vec{I} = \left(I_{\boldsymbol{\nu}_1}, I_{\boldsymbol{\nu}_2}, \dots, I_{\boldsymbol{\nu}_{N_{\mathrm{master}}}} \right)^T .$$

Master integrals

Summary:
We may write any Feynman integral from a family of Feynman integrals as a linear combination of the master integrals

$$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B}) = \sum_{j=1}^{N_{\mathrm{master}}} c_j \, I_{\boldsymbol{\nu}_j}(D, x_1, \dots, x_{N_B}),$$

where the coefficients c_j are rational functions of D and the kinematic variables x.

Section 4

Dimensional shift relations

Notation

Let's replace in the graph polynomials the (Schwinger/Feynman) parameters with the raising operators $\mathbf{j}^+$, i.e. we consider expressions

$$\mathcal{U}\left(\mathbf{1}^+, \dots, \mathbf{n_{\mathrm{int}}}^+\right) I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}.$$

Example: For $\mathcal{U}(\alpha_1, \alpha_2, \alpha_3) = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3$ we have

$$\begin{aligned} \mathcal{U}\left(\mathbf{1}^+, \mathbf{2}^+, \mathbf{3}^+\right) I_{111} &= \left(\mathbf{1}^+\mathbf{2}^+ + \mathbf{2}^+\mathbf{3}^+ + \mathbf{1}^+\mathbf{3}^+\right) I_{111} \\ &= I_{221} + I_{122} + I_{212}. \end{aligned}$$

Dimensional shift relations

Recall

$$\mathbf{D}^+ I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D) = \frac{e^{l \varepsilon \gamma_{\mathrm{E}}}}{\prod\limits_{k=1}^{n_{\mathrm{int}}} \Gamma(\nu_k)} \int\limits_{\alpha_k \geq 0} d^{n_{\mathrm{int}}}\alpha \left(\prod_{k=1}^{n_{\mathrm{int}}} \alpha_k^{\nu_k - 1} \right) \frac{1}{\mathcal{U} \cdot \mathcal{U}^{\frac{D}{2}}} e^{-\frac{\mathcal{F}}{\mathcal{U}}},$$

$$\mathbf{j}^+ I_{\nu_1 \dots \nu_j \dots \nu_{n_{\mathrm{int}}}}(D) = \frac{e^{l \varepsilon \gamma_{\mathrm{E}}}}{\prod\limits_{k=1}^{n_{\mathrm{int}}} \Gamma(\nu_k)} \int\limits_{\alpha_k \geq 0} d^{n_{\mathrm{int}}}\alpha \left(\prod_{k=1}^{n_{\mathrm{int}}} \alpha_k^{\nu_k - 1} \right) \frac{\alpha_{\mathbf{j}}}{\mathcal{U}^{\frac{D}{2}}} e^{-\frac{\mathcal{F}}{\mathcal{U}}}.$$

Thus

$$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D) = \mathcal{U}\left(\mathbf{1}^+, \dots, \mathbf{n}_{\mathrm{int}}{}^+\right) \mathbf{D}^+ I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D).$$

Dimensional shift relations

Dimensional shift relations:

$$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D) = \mathcal{U}\left(\mathbf{1}^+, \dots, \mathbf{n}_{\mathrm{int}}{}^+\right) I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D+2).$$

- Let $\vec{I} = (I_{\boldsymbol{\nu}_1}, \dots, I_{\boldsymbol{\nu}_{N_{\mathrm{master}}}})^T$ be a **basis in D space-time dimensions** and $\vec{I}' = (I'_{\boldsymbol{\nu}_1}, \dots, I'_{\boldsymbol{\nu}_{N_{\mathrm{master}}}})^T$ be a **basis in $(D+2)$ space-time dimensions**.
- Apply the shift relation to all integrals from $\vec{I}$ and reduce the integrals on the right-hand side with IBP-identities to $\vec{I}'$: We obtain a $(N_{\mathrm{master}} \times N_{\mathrm{master}})$-matrix S

$$\vec{I} = S \vec{I}'.$$

- Within dimensional regularisation the matrix S is invertible. Inverting this matrix allows us to express any master integral in $(D+2)$ dimensions as a linear combination of master integrals in D dimensions:

$$\vec{I}' = S^{-1} \vec{I}.$$

Example

Consider again the one-loop two-point function with equal internal masses:
We have

$$\begin{pmatrix} I_{10}(D,x) \\ I_{11}(D,x) \end{pmatrix} = \begin{pmatrix} -\frac{D}{2} & 0 \\ -\frac{D}{4+x} & -\frac{2(D-1)}{4+x} \end{pmatrix} \begin{pmatrix} I_{10}(D+2,x) \\ I_{11}(D+2,x) \end{pmatrix}$$

and

$$\begin{pmatrix} I_{10}(D+2,x) \\ I_{11}(D+2,x) \end{pmatrix} = \begin{pmatrix} -\frac{2}{D} & 0 \\ \frac{1}{D-1} & -\frac{4+x}{2(D-1)} \end{pmatrix} \begin{pmatrix} I_{10}(D,x) \\ I_{11}(D,x) \end{pmatrix}.$$

Section 5

Differential equations

The method of differential equations

Denote by $x = (x_1, ..., x_{N_B})$ the kinematic variables (scalar products of external momenta and internal masses squared).

We want to calculate

$$I_{\nu_1 ... \nu_{n_{\mathrm{int}}}}(D, x)$$

1. *Find a differential equation with respect to the kinematic variables for the Feynman integral (always possible).*
2. *Transform the differential equation into a simple form (**bottle neck**).*
3. *Solve the latter differential equation with appropriate boundary conditions (always possible).*

Subsection 1

Deriving the differential equation

Differential equations

Let x_k be a kinematic variable. Let $I_i \in \{I_1, ..., I_{N_{\mathrm{master}}}\}$ be a master integral. Carrying out the derivative

$$\frac{\partial}{\partial x_k} I_i$$

under the integral sign and using integration-by-parts identities allows us to express the **derivative as a linear combination of the master integrals**.

$$\frac{\partial}{\partial x_k} I_i = \sum_{j=1}^{N_F} a_{ij} I_j$$

Differential equations

The second Symanzik polynomial $\mathcal{F}$ is linear in the kinematic variables x_j. Set

$$\mathcal{F}'_{x_j}(a) \quad = \quad \frac{\partial}{\partial x_j} \mathcal{F}(a).$$

From the Schwinger parameter representation:

$$\frac{\partial}{\partial x_j} I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D, x) \quad = \quad -\mathcal{F}'_{x_j}\left(\mathbf{1}^+, \dots, \mathbf{n}_{\mathrm{int}}{}^+\right) I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D+2, x)$$

Differential equations

Let us formalise this:

$$I = (I_1, ..., I_{N_{\text{master}}}), \quad \text{set of master integrals,}$$
$$x = (x_1, ..., x_{N_B}), \quad \text{set of kinematic variables the master integrals depend on.}$$

We obtain a **system of differential equations**

$$dI + AI = 0,$$

where $A(\varepsilon, x)$ is a matrix-valued one-form

$$A = \sum_{i=1}^{N_B} A_i dx_i,$$

satisfying the integrability condition

$$dA + A \wedge A = 0.$$

Differential equations in ε-form

The system of differential equations is particular simple, if A is of the form

$$A = \varepsilon \sum_{j=1}^{N_L} C_j \, \omega_j ,$$

where

- C_j is a $N_{\mathrm{master}} \times N_{\mathrm{master}}$-matrix, whose entries are (rational or integer) numbers,
- the **only dependence on ε** is **given by** the **explicit prefactor**,
- the differential one-forms ω_j have **only simple poles**.

Subsection 2

Solving a differential equation in ε-form

Solving a differential equation in ε-form

Assume

1. The differential equation for $\vec{I}$ is in **ε-form**:

$$(d + A)\vec{I} = 0, \qquad A = \varepsilon \sum_{j=1}^{N_L} C_j \, \omega_j.$$

2. All master integrals have a **Taylor expansion** in ε:

$$I_{\nu_i}(\varepsilon, x) = \sum_{j=0}^{\infty} I_{\nu_i}^{(j)}(x) \cdot \varepsilon^j.$$

3. We know suitable **boundary values** for all master integrals.

Solving a differential equation in ε-form

We plug the Taylor expansion into the differential equation

$$\left(d + \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k \right) \left(\sum_{j=0}^{\infty} \vec{I}^{(j)}(x) \cdot \varepsilon^j \right) = 0,$$

and compare term-by-term in the ε-expansion.

We obtain

$$\begin{aligned} d\vec{I}^{(0)}(x) &= 0, \\ d\vec{I}^{(j)}(x) &= -\sum_{k=1}^{N_L} \omega_k \, C_k \, \vec{I}^{(j-1)}(x), \quad j \geq 1. \end{aligned}$$

Chen's iterated integrals

Definition

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \rightarrow M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda = \gamma^* \omega_j.$$

The **iterated integral** is defined by

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) = \int\limits_0^{\lambda} d\lambda_1 f_1(\lambda_1) \int\limits_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int\limits_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Multiple polylogarithms

We are interested in differential one-forms, which have only simple poles.
The simplest case:

$$\omega^{\mathrm{mpl}}(z_j) = \frac{d\lambda}{\lambda - z_j}.$$

Definition (Multiple polylogarithms)

$$G(z_1, \ldots, z_k; \lambda) = \int\limits_0^{\lambda} \frac{d\lambda_1}{\lambda_1 - z_1} \int\limits_0^{\lambda_1} \frac{d\lambda_2}{\lambda_2 - z_2} \ldots \int\limits_0^{\lambda_{k-1}} \frac{d\lambda_k}{\lambda_k - z_k}, \qquad z_k \neq 0$$

The method of differential equations

Example

One integral I in one variable x with **boundary condition $I(0) = 1$**. Consider the differential equation

$$(d + A) I = 0, \qquad A = -\varepsilon \frac{dx}{x-1}.$$

Then

$$I(x) = 1 + \varepsilon G(1;x) + \varepsilon^2 G(1,1;x) + \varepsilon^3 G(1,1,1;x) + \dots$$

Multiple polylogarithms

Definition based on **iterated integrals**:

$$G(z_1,...,z_k;y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} ... \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}$$

Definition based on **nested sums**:

$$\mathrm{Li}_{m_1,m_2,...,m_k}(x_1,x_2,...,x_k) = \sum_{n_1>n_2>...>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot ... \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Conversion:

$$\mathrm{Li}_{m_1,...,m_k}(x_1,...,x_k) = (-1)^k G_{m_1,...,m_k}\left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, ..., \frac{1}{x_1 ... x_k}; 1\right)$$

Short hand notation:

$$G_{m_1,...,m_k}(z_1,...,z_k;y) = G(\underbrace{0,...,0}_{m_1-1}, z_1,...,z_{k-1}, \underbrace{0...,0}_{m_k-1}, z_k; y)$$

Weights

Define the **weight** *of a multiple polylogarithm as*

$$\begin{aligned} \text{weight}\left(G_{m_1,\dots,m_k}(z_1,\dots,z_k;y)\right) &= m_1+\dots+m_k, \\ \text{weight}\left(\text{Li}_{m_1,\dots,m_k}(x_1,\dots,x_k)\right) &= m_1+\dots+m_k. \end{aligned}$$

If the differential equation is in ε*-form and all* ω_j*'s are of the form*

$$\omega_j = d\ln\left(p_j(x)\right),$$

where $p_j(x)$ *is a* **polynomial** *in the kinematic variables, then the master integrals can be expressed in terms of multiple polylogarithms and are of uniform weight.*