Feynman Integrals

Lecture 4

Stefan Weinzierl

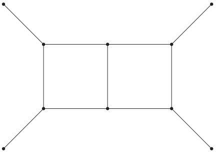
Institut für Physik, Universität Mainz

Higgs Centre School for Theoretical Physics 2021

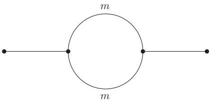
Section 1

Transformations of the differential equation

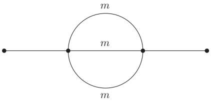
Examples



- Two-loop double box
  - 8 master integrals
  - 1 kinematic variable



- One-loop bubble
  - 2 master integrals
  - 1 kinematic variable



- Two-loop sunrise
  - 3 master integrals
  - 1 kinematic variable

Subsection 1

Fibre bundles

Fibre bundles

A fibre bundle consists of the following elements:

- A differentiable manifold E called the **total space**.
- A differentiable manifold M called the **base space**.
- A differentiable manifold F called the **fibre**.
- A **projection** $\pi : E \to M$. The inverse image $\pi^{-1}(p) = F_p$ is called the fibre at p.
- A Lie group G called the **structure group**, which acts on F from the left.

Principal bundles, vector bundles and connections

- A **principal bundle** P is a fibre bundle, whose fibre is identical with the structure group G.
- A **vector bundle** is a fibre bundle, whose fibre is a vector space. The dimension r of the fibre F is called the **rank** of the vector bundle.
- A **connection one-form** ω, which takes values in the Lie algebra $\mathfrak{g}$ of G, is a projection of $T_u P$ onto the vertical component $V_u P \cong \mathfrak{g}$, such that the horizontal subspaces $H_u P$ and $H_{ug} P$ on the same fibre are related by a linear map induced by $g \in G$.
- Denote by ***A*** the **pull-back** of ω by a section $s : M \to P$ to M:

$$A = s^* \omega.$$

A defines a **covariant derivative**:

$$\nabla = d + A.$$

Fibre bundles in physics

- **Quarks (QCD)**

Base space:	Minkowski space
Fibre:	3-dimensional vector space
Local connection one-form:	$A = \frac{g}{i} T^a A^a_\mu dx^\mu$

- **General relativity**

Base space:	(curved) space-time
Fibre:	Metric
Local connection one-form:	Levi-Civita connection

Feynman integrals

We have a vector bundle:

- Fibre spanned by the master integrals $I_{\nu_1}, ..., I_{\nu_{N_{\mathrm{master}}}}$.

  (The master integrals $I_{\nu_1}(x), \dots, I_{\nu_{N_{\mathrm{master}}}}(x)$ can be viewed as local sections, and for each x they define a basis of the vector space in the fibre.)
- Base space with coordinates $x = (x_1, ..., x_{N_B})$ corresponding to kinematic variables.
- Connection defined by the matrix A.

We would like to **transform** this vector bundle **to a standard form** through

- a change of basis in the fibre,
- a coordinate transformation on the base manifold.

Transformations

- **Change the basis of the master integrals**

$$\vec{I}' = U\vec{I},$$

where $U(\varepsilon, x)$ is a $N_{\mathrm{master}} \times N_{\mathrm{master}}$-matrix. The new connection matrix is

$$A' = UAU^{-1} + UdU^{-1}.$$

- **Perform a coordinate transformation on the base manifold**:

$$x_i' = f_i(x), \qquad 1 \leq i \leq N_B.$$

The connection transforms as

$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x_j'} dx_j'.$$

Subsection 2

Fibre transformations

Overview

We seek a transformation $\vec{I}' = U\vec{I}$ such that $A' = UAU^{-1} + UdU^{-1}$ is simpler.

- **Block decomposition**
- Reduction to an univariate problem
- Picard-Fuchs operators
- Exploitung a master integral known to be of uniform weight
- Magnus expansion
- Moser's algorithm
- Leinartas decomposition
- **Maximal cuts and constant leading singularities**

Block decomposition

Order the set of master integrals $\vec{I} = (I_{\boldsymbol{\nu}_1}, \dots, I_{\boldsymbol{\nu}_{N_{\mathrm{master}}}})^T$ such that $I_{\boldsymbol{\nu}_1}$ is the simplest integral and $I_{\boldsymbol{\nu}_{N_{\mathrm{master}}}}$ the most complicated integral.

The matrix A has a lower block-triangular structure:

$$A = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ A_3 & \multicolumn{2}{c}{A_2} & \begin{matrix} 0 \\ 0 \end{matrix} \\ A_6 & \multicolumn{2}{c}{A_5} & A_4 \end{pmatrix}$$

Diagonal blocks: A_1, A_2, A_4

Non-diagonal blocks: A_3, A_5, A_6

Diagonal blocks

Let's consider block A_2. We consider a transformation of the form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_2^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transformed A' is given by

$$A' = \begin{pmatrix} A_1 & 0 & 0 \\ U_2 A_3 & \mathbf{U_2 A_2 U_2^{-1} + U_2 d U_2^{-1}} & 0 \\ A_6 & A_5 U_2^{-1} & A_4 \end{pmatrix}.$$

Suppose the block A_2 contains an unwanted term F and a remainder R:

$$A_2 = F + R.$$

The term F can be removed by a fibre transformation with U_2 given as a solution of the differential equation

$$dU_2^{-1} = -F U_2^{-1}.$$

Example

Assume that we have only one kinematic variable x_1 (e.g. $N_B = 1$) and that A_2 is of size (1×1) and given by

$$A_2 = \left(\frac{1}{x-1} + \frac{2\varepsilon}{x-1} \right) dx.$$

We would like to remove the first term $F = dx/(x-1)$ by a fibre transformation. We have to solve the differential equation

$$\frac{d}{dx} U_2^{-1} + \frac{1}{x-1} U_2^{-1} = 0.$$

A solution is easily found and given by

$$U_2^{-1} = \frac{C}{x-1}, \qquad U_2 = C^{-1} \left(x-1\right).$$

We may set $C = 1$ and $U_2 = x - 1$ is the sought-after transformation.

Non-diagonal blocks

Let us now consider block A_3. At this stage we would like to preserve the blocks A_1 and A_2. We consider a transformation of the form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ U_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -U_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transformed A' is given by

$$A' = \begin{pmatrix} A_1 & 0 & 0 \\ \mathbf{A_3 - A_2 U_3 + U_3 A_1 - dU_3} & A_2 & 0 \\ A_6 - A_5 U_3 & A_5 & A_4 \end{pmatrix}.$$

Suppose the block A_3 contains an unwanted term F and a remainder R:

$$A_3 = F + R.$$

The term F can be removed by a fibre transformation with U_3 given as a solution of the differential equation

$$dU_3 + A_2 U_3 - U_3 A_1 = F.$$

Example

We again consider the case of one kinematic variable x (e.g. $N_B = 1$). We further assume that A_1 and A_2 are both blocks of size (1×1). Then A_3 is also a block of size (1×1). Assume that A_1 and A_2 are already in ε-form an given by

$$A_1 = \frac{\varepsilon dx}{x-1}, \qquad A_2 = \frac{2\varepsilon dx}{x-1}.$$

Assume further that F is given by

$$F = \frac{\mathbf{dx}}{(\mathbf{x}-\mathbf{1})^{\mathbf{2}}}.$$

We have to solve the differential equation

$$\left[\frac{d}{dx} + \frac{\varepsilon}{x-1} \right] U_3 = \frac{1}{(x-1)^2}.$$

A solution is given by

$$U_3 = \frac{1}{(1-\varepsilon)(1-x)}.$$

Quiz

Let $\vec{I} = (I_1, I_2, I_3)^T$ be a set of master integrals with

$$(d + A)\vec{I} = 0, \quad A = \varepsilon \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 3 & 7 \end{pmatrix} \left(\frac{dx}{x} - \frac{2dx}{x-1} \right)$$

and

$$\begin{aligned} I'_1 &= I_1, \\ I'_2 &= I_2, \\ I'_3 &= 42 I_3 - I_2 - I_1. \end{aligned}$$

Which of the following statements is true?

(A) A' is again in ε-form.

(B) The non-diagonal entries of A' are not in ε-form.

(C) Only the sub-system of the first two master integrals is in ε-form.

(D) $\vec{I}'$ is not a basis of master integrals.

Subsection 3

Maximal cuts and constant leading singularities

Heuristic methods

- Suppose somebody gives us a transformation matrix U

$$\vec{I}' = U\vec{I}.$$

- It is easy to check if this fibre transformation transforms the differential equation to an ε-form. We simply calculate

$$A' = UAU^{-1} + UdU^{-1}$$

  and check if A' is in ε-form.

- This is a situation where a heuristic method may work well: Guessing a suitable U may outperform any systematic algorithm to construct the matrix U.

Feynman integrals with cuts

Recal: Baikov representation

$$I_{\nu_1 \dots \nu_n}(D, x_1, \dots, x_{N_B}) = C \int\limits_{\mathcal{C}} d^{N_V} z \, [\mathcal{B}(z)]^{\frac{D-l-e-1}{2}} \prod_{s=1}^{N_V} z_s^{-\nu_s}$$

with integration contour $\mathcal{C}$.

Consider a **modified integration contour** $\mathcal{C}'$ such that

1. Integration-by-parts identities still hold.
2. The variation of the integral with respect to the kinematic variables comes entirely from the integrand.
3. The symmetries among the integrals are respected.

The maximal cut

Definition (Feynman integral with the internal edge e_j cut)

Baikov integral with a modified integration domain $\mathcal{C}'$:

- a small anti-clockwise circle around $z_j = 0$ in the complex z_j-plane,
- in all other variables the intersection of the original integration domain $\mathcal{C}$ with the hyperplane $z_j = 0$.

We may iterate the procedure and take multiple cuts. Of particular importance is the maximal cut:

Definition (Maximal cut)

Take for a Feynman integral $I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}$ the cut for all edges e_j for which $\nu_j > 0$.

Example

One-loop two-point function with equal internal masses:
Baikov polynomial ($x = -p^2/m^2$ and $\mu^2 = m^2 = 1$):

$$\mathcal{B}(z_1, z_2) = -\frac{1}{4}\left[(z_1 - z_2)^2 - 2x(z_1 + z_2) + x(4+x)\right],$$

Baikov representation of I_{11}:

$$I_{11} = \frac{e^{\varepsilon\gamma_{\mathrm{E}}} x^{-\frac{D-2}{2}}}{2\sqrt{\pi}\Gamma\left(\frac{D-1}{2}\right)} \int_{\mathcal{C}} d^2z \left[\mathcal{B}(z_1, z_2)\right]^{\frac{D-3}{2}} \frac{1}{z_1 z_2}.$$

Maximal cut:

$$\mathrm{MaxCut}\, I_{11} = (2\pi i)^2 \frac{e^{\varepsilon\gamma_{\mathrm{E}}} x^{-\frac{D-2}{2}}}{2\sqrt{\pi}\Gamma\left(\frac{D-1}{2}\right)} \left(-\frac{1}{4}x(4+x)\right)^{\frac{D-3}{2}}.$$

In $D = 2 - 2\varepsilon$ dimensions we have to leading order in the ε-expansion:

$$\mathrm{MaxCut}\, I_{11}(2-2\varepsilon) = -\frac{4\pi}{\sqrt{-x(4+x)}} + \mathcal{O}(\varepsilon).$$

Constant leading singularities

- Denote the **integrands** of the master integrals by $\varphi_1, \dots, \varphi_{N_{\mathrm{master}}}$.
- Choose N_{master} **independent integration domains** $\mathcal{C}_1, \dots, \mathcal{C}_{N_{\mathrm{master}}}$. The integration domains are independent, if the $N_{\mathrm{master}} \times N_{\mathrm{master}}$-matrix with entries

$$\langle \varphi_i | \mathcal{C}_j \rangle = \int\limits_{\mathcal{C}_j} \varphi_i$$

  has full rank.
- We are interested in choosing the integration domains $\mathcal{C}_j$ as simple as possible. Particular simple integration domains are products of circles around singular points. These correspond to residue calculations.

Constant leading singularities

- Let φ be the integrand of a Feynman integral I.
- Define d_{min} by

$$d_{\mathrm{min}} = \min_j \left(\mathrm{ldegree}\left(\langle \varphi | \mathcal{C}_j \rangle, \varepsilon \right) \right),$$

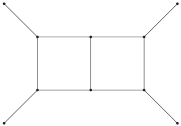
- We say that the Feynman integral I has **constant leading singularities**, if for all j

$$\mathrm{coeff}\left(\langle \varphi | \mathcal{C}_j \rangle, \varepsilon^{d_{\mathrm{min}}} \right) = \text{constant of weight zero},$$

- Integrals with constant leading singularities are a guess for a basis of master integrals, which puts the differential equation into an ε-form.

Example

- Consider the two-loop double box integral with vanishing internal masses, $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$ and $x = s/t$.



- This is a system with eight master integrals.
- Suppose we already found suitable master integrals, which puts the sub-system of the first six master integrals into an ε-form.
- Thus we are left with finding a fibre transformation, which transforms the **last sector**, consisting of the two master integrals $I_{111111100}$ and $I_{1111111(-1)0}$ into an ε-form.

Example

Consider the maximal cut of this sector for the integrals $I_{1111111\nu 0}$.
With $\mu^2 = t$ we have

$$\text{MaxCut}\, I_{1111111\nu 0} = (2\pi i)^7 \frac{2^{4\varepsilon}(s+t)^{\varepsilon}\, t^{3+\nu+3\varepsilon}}{4\pi^3 \left(\Gamma\left(\frac{1}{2}-\varepsilon\right)\right)^2 s^{2+2\varepsilon}} \int\limits_{\mathcal{C}_{\text{MaxCut}}} dz_8\, z_8^{-1-2\varepsilon} (t-z_8)^{-1-\varepsilon} (s+t-z_8)^{\varepsilon}\, z_8^{-\nu}.$$

We now choose two independent integration domains:

$$\begin{aligned} \mathcal{C}_1 &: \text{ small circle around } z_8 = 0 \text{ for the } z_8\text{-integration,} \\ \mathcal{C}_2 &: \text{ small circle around } z_8 = t \text{ for the } z_8\text{-integration.} \end{aligned}$$

We set

$$\varphi_\nu = \frac{2^{4\varepsilon}(s+t)^{\varepsilon}\, t^{3+\nu+3\varepsilon}}{4\pi^3 \left(\Gamma\left(\frac{1}{2}-\varepsilon\right)\right)^2 s^{2+2\varepsilon}}\, z_8^{-1-2\varepsilon} (t-z_8)^{-1-\varepsilon} (s+t-z_8)^{\varepsilon}\, z_8^{-\nu}\, d^8 z.$$

Example

With $x = s/t$ we have

$$\langle \varphi_0 | \mathcal{C}_1 \rangle = \frac{64\pi^4}{x^2} + \mathcal{O}(\varepsilon), \qquad \langle \varphi_0 | \mathcal{C}_2 \rangle = -\frac{64\pi^4}{x^2} + \mathcal{O}(\varepsilon).$$

The integral

$$\mathrm{MaxCut}\ I_{111111100} = \langle \varphi_0 | \mathcal{C}_{\mathrm{MaxCut}} \rangle$$

does not have constant leading singularities, but it is **easy to fix** this issue:

- We multiply the integrand by x^2.
- If in addition we multiply by ε^4, the leading singularities are constants of weight zero.
- Strictly speaking we can only infer from the first term of the ε-expansion of $\langle \varphi_0 | \mathcal{C}_j \rangle$ that we should multiply by an ε-dependent prefactor, whose ε-expansion starts at ε^4. In this example we can verify a posteriori that ε^4 is the correct ε-dependent prefactor.

Example

Set

$$\varphi_0' = \varepsilon^4 x^2 \varphi_0.$$

Then

$$\left\langle \varphi_0' \middle| \mathcal{C}_1 \right\rangle = 64\pi^4\varepsilon^4 + \mathcal{O}(\varepsilon), \qquad \left\langle \varphi_0' \middle| \mathcal{C}_2 \right\rangle = -64\pi^4\varepsilon^4 + \mathcal{O}(\varepsilon).$$

Thus

$$\mathrm{MaxCut}\left(\varepsilon^4 x^2 I_{111111100}\right) = \left\langle \varphi_0' \middle| \mathcal{C}_{\mathrm{MaxCut}} \right\rangle$$

has constant leading singularities.

Example

As this sector has two master integrals, we **need a second master integral**.
We consider φ_{-1} and compute the leading singularities. We obtain

$$\langle \varphi_{-1} | \mathcal{C}_1 \rangle = 0 + \mathcal{O}(\varepsilon), \qquad \langle \varphi_{-1} | \mathcal{C}_2 \rangle = -\frac{64\pi^4}{x^2} + \mathcal{O}(\varepsilon).$$

It follows that

$$\mathrm{MaxCut}\left(\varepsilon^4 x^2 I_{1111111(-1)0}\right) = \left\langle \varepsilon^4 x^2 \varphi_{-1} | \mathcal{C}_{\mathrm{MaxCut}} \right\rangle$$

has constant leading singularities.

Example

It is easily verified, that the two master integrals

$$\varepsilon^4 x^2 I_{111111100} \quad \text{and} \quad \varepsilon^4 x^2 I_{1111111(-1)0}$$

put the **2×2-diagonal block** for this sector into an **ε-form**.

It remains to treat the off-diagonal block with entries $A_{i,j}$, $i \in \{7,8\}$, $j \in \{1,2,3,4,5,6\}$. This is most easily done with the methods discussed in the context of block decomposition. One finds

$$\begin{aligned} I'_{\nu_7} &= \varepsilon^4 x^2 I_{111111100}, \\ I'_{\nu_8} &= \varepsilon^4 x^2 I_{1111111(-1)0} + x \left[I'_{\nu_6} + \frac{1}{2} \left(I'_{\nu_5} + I'_{\nu_4} - I'_{\nu_2} - I'_{\nu_1} \right) \right]. \end{aligned}$$

Subsection 4

Base transformations

Base transformations

Coordinate transformation on the base manifold:

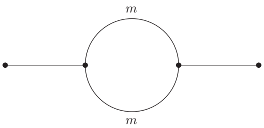
$$x_i' = f_i(x), \qquad 1 \le i \le N_B.$$

The connection transforms as

$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x_j'} dx_j'.$$

Example

The one-loop two point function:



Master integrals:

$$\vec{I} = \begin{pmatrix} I_{10} \\ I_{11} \end{pmatrix}$$

Differential equation:

$$(d + A)\vec{I} = 0, \qquad A = \begin{pmatrix} 0 & 0 \\ \frac{1-\varepsilon}{2x} - \frac{1-\varepsilon}{2(x+4)} & \frac{1}{2x} - \frac{1-2\varepsilon}{2(x+4)} \end{pmatrix} dx.$$

Example

There is no fibre transformation **rational** in x and ε, which factors out ε. However, if we allow the transformation to be **algebraic**, we may achieve this goal.

$$\vec{I}' = U \vec{I}, \qquad U = \left(\begin{array}{cc} 2\varepsilon\left(1-\varepsilon\right) & 0 \\ 2\varepsilon\left(1-\varepsilon\right)\sqrt{\frac{x}{4+x}} & 2\varepsilon\left(1-2\varepsilon\right)\sqrt{\frac{x}{4+x}} \end{array} \right).$$

For the transformed system we find

$$\left(d + A'\right) \vec{I}' = 0, \qquad A' = \varepsilon \left(\begin{array}{cc} 0 & 0 \\ -\frac{dx}{\sqrt{x(4+x)}} & \frac{dx}{4+x} \end{array} \right).$$

Example

We have achieved that ε only appears as a prefactor, however the condition that the only singularities are **simple poles** is not met: The differential one-form

$$\frac{dx}{\sqrt{x(4+x)}}$$

has square root singularities at $x = 0$ and $x = -4$.

Remark:

$$\frac{dx}{\sqrt{x(4+x)}} = d \ln\left(2 + x + \sqrt{x(4+x)}\right).$$

We see that in this case the argument of the logarithm is no longer a polynomial, but an **algebraic function** of x.

Example

Let's **define** x' by

$$x = \frac{(1-x')^2}{x'}.$$

The inverse relation reads

$$x' = \frac{1}{2}\left(2+x-\sqrt{x(4+x)}\right),$$

where we made a choice for the sign of the square root. We have

$$\frac{\partial x}{\partial x'} = -\frac{(1-x')^2}{x'^2}$$

and

$$\frac{dx}{\sqrt{x(4+x)}} = -\frac{dx'}{x'}, \qquad \frac{dx}{4+x} = \frac{2dx'}{x'+1} - \frac{dx'}{x'}.$$

Example

Thus in term of the new variable x' we have

$$\left(d + A'\right)\vec{I}' = 0, \qquad A' = \varepsilon \begin{pmatrix} 0 & 0 \\ \frac{dx'}{x'} & \frac{2dx'}{x'+1} - \frac{dx'}{x'} \end{pmatrix}.$$

The differential equation is now in **ε-form**:

- The dimensional regularisation parameter occurs only as a prefactor
- The only singularities of A' are simple poles.
- For the case at hand, A' has simple poles at $x' = 0$ and $x' = -1$.

Rationalising square roots

Consider

$$\sqrt{f(x_1,\dots,x_n)} \quad \text{and} \quad V(f) = \{x \in \mathbb{C}^n | f(x) = 0\}.$$

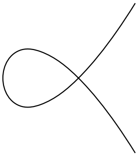
A point $p \in V$ is said to be of **multiplicity** $o \in \mathbb{N}$ if all partial derivatives of order $< o$ vanish at p

$$\frac{\partial^{i_1+\dots+i_n} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(p) = 0 \qquad \text{with } i_1 + \dots + i_n < o$$

and if there exists at least one non-vanishing o-th partial derivative

$$\frac{\partial^{i_1+\dots+i_n} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(p) \neq 0 \qquad \text{with } i_1 + \dots + i_n = o.$$

Rationalising square roots



Points of multiplicity 1 are called **regular points**,
points of multiplicity $o > 1$ are called **singular points** of V.

Theorem

Let $f(x_1, \dots, x_n)$ be a polynomial of degree d. If $V(f)$ has a point of multiplicity $(d-1)$, the square root $\sqrt{f(x_1, \dots, x_n)}$ can be rationalised.

Section 2

Intersection numbers

Integration by parts

Integration by parts:
We may write any Feynman integral from a family of Feynman integrals as a linear combination of the master integrals

$$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B}) = \sum_{j=1}^{N_{\mathrm{master}}} c_j \, I_{\boldsymbol{\nu}_j}(D, x_1, \dots, x_{N_B}),$$

where the coefficients c_j are rational functions of D and the kinematic variables x.

Questions:

- Is there a scalar product on the vector space of Feynman integrals?
- Can we use this scalar product for the reduction to master integrals?

$$\vec{a} = \sum_j c_j \vec{e}_j \quad \Rightarrow \quad c_j = \vec{a} \cdot \vec{e}_j$$

Intersection numbers

- In the following we will consider **integrands** of Feynman integrals instead of Feynman integrals.
- We will work in the Baikov representation.
- The integrand will be a differential N_V-form in the Baikov variables z, where N_V denotes the number of Baikov variables.
- Number of independent integrands: N_{cohom}
  Number of master integrals: N_{master}

Recap: The Baikov representation

$$I_{\nu_1\nu_2\dots\nu_n} = \text{const} \int\limits_{\mathcal{C}} \underbrace{B^{\frac{D-l-e-1}{2}}}_{\text{multi-valued function}} \underbrace{\prod_{s=1}^{n} z_s^{-\nu_s}\, d^n z}_{\text{differential n-form}}$$

Notation in the following:

multi-valued function	$u = B^{\frac{D-l-e-1}{2}},$
differential one-form	$\omega = d \ln u,$
covariant derivative	$\nabla_\omega = d + \omega,$
differential n-form	$\varphi = \prod_{s=1}^{n} z_s^{-\nu_s}\, d^n z,$
divisor	$\text{Div} = \{B = 0\} \subset \mathbb{C}^n \qquad (1)$

Twisted periods

Let us now consider integrals of the form

$$\mathbf{I} \;=\; \int\limits_{\mathcal{C}} \mathbf{u}\,\boldsymbol{\varphi}, \qquad u \;=\; \prod_{i=1}^{m} p_i^{\gamma_i}, \qquad \omega \;=\; d \ln u \;=\; \sum_{j=1}^{n} \omega_j dz_j,$$

where the exponents γ_i are generic, in particular $\gamma_i \notin \mathbb{Z}$. Set

$$\mathrm{Div} \;=\; \bigcup_{i=1}^{m} \{p_i = 0\} \;\subset\; \mathbb{C}^n$$

φ is a rational holomorphic n-form on $\mathbb{C}^n - \mathrm{Div}$, e.g. of the form

$$\boldsymbol{\varphi} \;=\; \frac{\mathbf{q}}{\mathbf{p_1^{n_1} \dots p_m^{n_m}}}\,\mathbf{dz_n} \wedge \dots \wedge \mathbf{dz_1}, \qquad q \in \mathbb{K}\left[z_1, \dots, z_n\right], \quad n_i \in \mathbb{N}_0.$$

Integration-by-parts revisited

Consider integration cycles $\mathcal{C}$ such that u **vanishes on the integration boundary**, hence

$$\int\limits_{\partial \mathcal{C}} u\,\xi = \int\limits_{\mathcal{C}} d\left(u\,\xi\right) \;=\; 0.$$

This is equivalent to

$$\int\limits_{\mathcal{C}} u\,\left(\nabla_{\omega}\xi\right) \;=\; 0, \qquad \omega \;=\; d \ln u,$$

for any $(n-1)$-form ξ.

Twisted cohomology

Two n-forms φ' and φ are called **equivalent**, if

$$\varphi' \sim \varphi \quad \Leftrightarrow \quad \varphi' = \varphi + \nabla_\omega \xi.$$

Denote equivalence classes by $\langle \varphi |$. Each φ is trivially closed:

$$\nabla_\omega \varphi = 0.$$

The equivalence classes define the **twisted cohomology group** H^n_ω:

$$\langle \varphi | \in H^n_\omega = \frac{\nabla_\omega - \text{closed } n - \text{forms}}{\nabla_\omega - \text{exact } n - \text{forms}}.$$

Dual twisted cohomology

We may also consider the dual twisted cohomology group.

$$\varphi = \frac{q}{p_1^{n_1} \dots p_m^{n_m}} dz_1 \wedge \dots \wedge dz_n, \qquad q \in \mathbb{K}[z_1, \dots, z_n], \quad n_i \in \mathbb{N}_0.$$

Equivalence classes are now defined by

$$\left|\varphi'\right\rangle = \left|\varphi\right\rangle \quad \Leftrightarrow \quad \varphi' = \varphi + \nabla_{-\omega}\xi$$

Equivalence classes $\left|\varphi\right\rangle$ are elements of the dual twisted cohomology group

$$\left(H^n_\omega\right)^* = H^n_{-\omega}.$$

Key properties

1. The cohomology groups H^n_ω and $(H^n_\omega)^*$ are **finite-dimensional**.
2. There is a non-degenerate inner product between H^n_ω and $(H^n_\omega)^*$, called the **intersection number** and denoted by

$$\langle \varphi_L | \varphi_R \rangle, \qquad \langle \varphi_L | \in H^n_\omega, \qquad | \varphi_R \rangle \in (H^n_\omega)^*.$$

In particular: Set $\nu_n = \dim H^n_\omega = \dim (H^n_\omega)^*$ and let $\langle e_1^{(n)}|, \langle e_2^{(n)}|, \ldots$ be a basis of H^n_ω. Then there is a dual basis $|d_1^{(n)}\rangle, |d_2^{(n)}\rangle, \ldots$ of $(H^n_\omega)^*$ with

$$\left\langle e_j^{(n)} \middle| d_k^{(n)} \right\rangle = \delta_{jk}.$$

Application

Let $\langle e_1^{(n)}|, \langle e_2^{(n)}|, ...$ be a basis of H_ω^n and consider an arbitrary element $\langle \varphi_L| \in H_\omega^n$.
We may **express $\langle \varphi_L|$ in terms of the basis**:

$$\langle \varphi_L| = c_1 \left\langle e_1^{(n)} \right| + c_2 \left\langle e_2^{(n)} \right| + ...$$

The **coefficients** are **given by the intersection numbers**

$$c_j = \left\langle \varphi_L \middle| d_j^{(n)} \right\rangle$$

This is exactly what we need for Feynman integral reduction!

Intersection numbers

Definition of the intersection number:

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)^n} \int \iota_\omega(\varphi_L) \wedge \varphi_R = \frac{1}{(2\pi i)^n} \int \varphi_L \wedge \iota_{-\omega}(\varphi_R),$$

where ι_ω maps φ_L to its **compactly supported** version, and similar for $\iota_{-\omega}$.

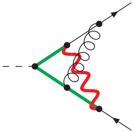
Recursive computation of intersection numbers

- In order to compute a multi-variate intersection number in n variables $z_1, \dots, z_n$, consider a sequence of fibrations $i \in \{0, 1, 2, \dots, n\}$, where we only compute the intersection number in the first i variables $z_1, \dots, z_i$.
- Suppose the intersection in the first $(i-1)$ variables has already been computed. Then we only need the univariate intersection in the variable z_i.
- Reduction to simple poles: Replace a representative of an equivalence class with higher poles with an equivalent representative with only simple poles.
- Evaluation of the intersection number as an univariate global residue. This is easily computed and does not involve algebraic extensions.

Example

Two-loop $\alpha\alpha_s$-corrections to Higgs decay:

$$I_{111111(-1)} = c\, I_{1111110} + ...,$$



where the dots stand for terms proportional to master integrals in lower sectors.

$$p_1 = \frac{1}{16}\left(z_7 - p^2\right)^2\left(z_7 + m_W^2 - m_t^2\right)^2, \quad u = p_1^{-\frac{1}{2}-\varepsilon}, \quad \omega = d\ln u.$$

z_7 is an auxiliary propagator. Basis / dual basis:

$$\hat{e}^{(1)}_{1111110} = 1, \qquad \hat{d}^{(1)}_{1111110} = \frac{2\left(1+4\varepsilon\right)\left(3+4\varepsilon\right)}{\left(1+2\varepsilon\right)\left(p^2+m_W^2-m_t^2\right)^2},$$

The sought-after coefficient c is then given by

$$\mathbf{c} = \left\langle \mathbf{z_7} \middle| \mathbf{d^{(1)}_{1111110}} \right\rangle = \frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{p^2} + \mathbf{m_t^2} - \mathbf{m_W^2}\right)$$