Feynman Integrals

Lecture 5

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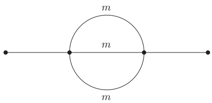
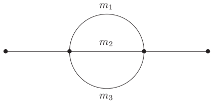
Section 1

Elliptic Feynman integrals

Motivation



- Two-loop equal mass sunrise
  - 3 master integrals
  - 1 kinematic variable



- Two-loop unequal mass sunrise
  - 7 master integrals
  - 3 kinematic variable

The equal mass sunrise

With $\vec{I} = (I_{110}, I_{111}, I_{211})^T$, $x = -p^2/m^2$ and $\mu^2 = m^2$ we have the differential equation $(d+A)\vec{I} = 0$ with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(D-3) & -3 \\ 0 & \frac{1}{6}(D-3)(3D-8) & \frac{1}{2}(3D-8) \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{8}(D-3)(3D-8) & -(D-3) \end{pmatrix} \frac{dx}{x+1} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{24}(D-3)(3D-8) & -(D-3) \end{pmatrix} \frac{dx}{x+9}.$$

Subsection 1

Background from Mathematics

Algebraic curves

- Ground field $\mathbb{C}$
- **Algebraic curve** in $\mathbb{C}^2$ **defined by** a **polynomial** $P(x,y)$:

$$P(x,y) = 0$$

- Projective space $\mathbb{CP}^2$ with homogeneous coordinates $[x:y:z]$:
  Algebraic curve in $\mathbb{CP}^2$ defined by a homogeneous polynomial $P(x,y,z)$:

$$P(x,y,z) = 0$$

  We usually work in the chart $z=1$.

Elliptic curves

Definition (Elliptic curve over $\mathbb{C}$)

An algebraic curve in $\mathbb{CP}^2$ of genus one with one marked point.

Example (Weierstrass normal form)

In the chart $z = 1$:

$$y^2 = 4x^3 - g_2 x - g_3$$

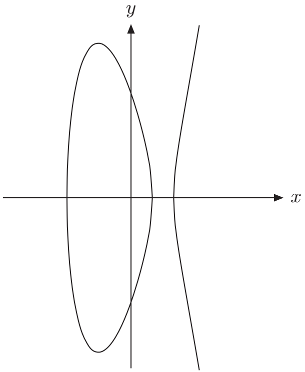
Example (Quartic form)

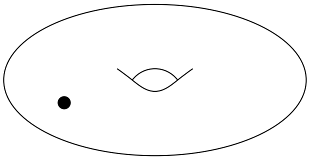
In the chart $z = 1$:

$$y^2 = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

Riemann surfaces

One complex dimension corresponds to two real dimensions.





Weierstrass normal form
$y^2 = 4x^3 - g_2 x - g_3$

Real Riemann surface of genus one with one marked point

Periodic functions

Let us consider a non-constant meromorphic function f of a complex variable z.

A **period** ψ of the function f is a constant such that for all z:

$$f(z+\psi) = f(z)$$

The set of all periods of f forms a **lattice**, which is either

- trivial (i.e. the lattice consists of $\psi = 0$ only),
- a simple lattice, $\Lambda = \{ n\psi \mid n \in \mathbb{Z} \}$,
- a double lattice, $\Lambda = \{ n_1\psi_1 + n_2\psi_2 \mid n_1, n_2 \in \mathbb{Z} \}$.

Double periodic functions are called **elliptic functions**.

Examples of periodic functions

- Singly periodic function: **Exponential function**

$$\exp(z).$$

$\exp(z)$ is periodic with peridod $\psi = 2\pi i$.

- Doubly periodic function: **Weierstrass's $\wp$-function**

$$\wp(z) = \frac{1}{z^2} + \sum_{\psi \in \Lambda \backslash \{0\}} \left(\frac{1}{(z+\psi)^2} - \frac{1}{\psi^2} \right), \qquad \Lambda = \{ n_1 \psi_1 + n_2 \psi_2 | n_1, n_2 \in \mathbb{Z} \},$$

$$\mathrm{Im}(\psi_2/\psi_1) \neq 0.$$

$\wp(z)$ is periodic with periods ψ_1 and ψ_2.

Inverse functions

The corresponding inverse functions are in general **multivalued functions**.

- For the exponential function $x = \exp(z)$ the inverse function is the **logarithm**

$$z = \ln(x).$$

- For Weierstrass's elliptic function $x = \wp(z)$ the inverse function is an **elliptic integral**

$$z = \int\limits_{x}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad g_2 = 60 \sum_{\psi \in \Lambda \backslash \{0\}} \frac{1}{\psi^4}, \quad g_3 = 140 \sum_{\psi \in \Lambda \backslash \{0\}} \frac{1}{\psi^6}.$$

Elliptic integrals

Complete elliptic integrals

- First kind:

$$K(x) = \int\limits_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}$$

- Second kind:

$$E(x) = \int\limits_0^1 dt \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}$$

- Third kind:

$$\Pi(v,x) = \int\limits_0^1 \frac{dt}{(1-vt^2)\sqrt{(1-t^2)(1-x^2t^2)}}$$

Incomplete elliptic integrals

- First kind:

$$F(z,x) = \int\limits_0^z \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}$$

- Second kind:

$$E(z,x) = \int\limits_0^z dt \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}$$

- Third kind:

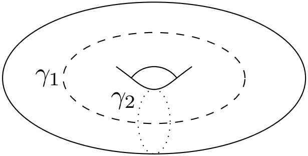
$$\Pi(v,z,x) = \int\limits_0^z \frac{dt}{(1-vt^2)\sqrt{(1-t^2)(1-x^2t^2)}}$$

Abelian differentials

- Abelian differential of the first kind:
  holomorphic
- Abelian differential of the second kind:
  meromorphic with **all residues vanishing**
- Abelian differential of the third kind:
  meromorphic with **non-zero residues**

Periods of an elliptic curve

Integrate the **holomorphic differential** along the two independent cycles.



Periods of an elliptic curve

Example

The Legendre form:

$$y^2 = x(x-1)(x-\lambda)$$

The periods are

$$\psi_1 = 2 \int\limits_0^\lambda \frac{dx}{y} = 4K\left(\sqrt{\lambda}\right) \qquad \psi_2 = 2 \int\limits_1^\lambda \frac{dx}{y} = 4iK\left(\sqrt{1-\lambda}\right)$$

Picard-Fuchs operator

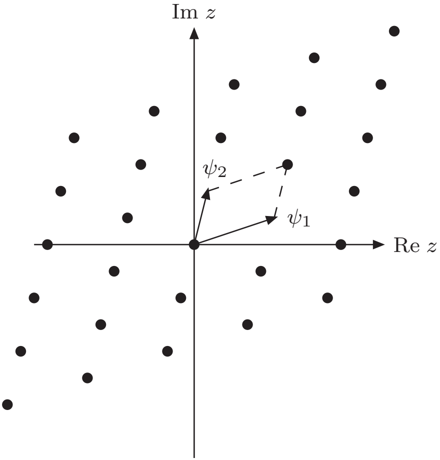
The elliptic curve $y^2 = x(x-1)(x-\lambda)$ depends on a parameter λ, and so do the periods $\psi_1(\lambda)$ and $\psi_2(\lambda)$.

How do the periods change, if we change λ?

The variation is governed by a second-order differential equation:
With $t = \sqrt{\lambda}$ we have

$$\underbrace{\left[t\left(1-t^2\right)\frac{d^2}{dt^2} + \left(1-3t^2\right)\frac{d}{dt} - t \right]}_{\text{Picard-Fuchs operator}} \psi_j \quad = \quad 0$$

Representing an elliptic curve as $\mathbb{C}/\Lambda$



Points inside fundamental parallelogram $\Leftrightarrow$ Points on elliptic curve

Back and forth

- **Weierstrass normal form** $\rightarrow \mathbb{C}/\Lambda$:

  Given a point (x, y) with $y^2 - 4x^3 + g_2 x + g_3 = 0$ the corresponding point $z \in \mathbb{C}/\Lambda$ is given by

$$z = \int\limits_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}$$

- $\mathbb{C}/\Lambda \rightarrow$ **Weierstrass normal form**:

  Given a point $z \in \mathbb{C}/\Lambda$ the corresponding point (x, y) on $y^2 - 4x^3 + g_2 x + g_3 = 0$ is given by

$$(x, y) = \left(\wp(z), \wp'(z)\right)$$

Notation

Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$, where

$$\tau = \frac{\psi_2}{\psi_1}$$

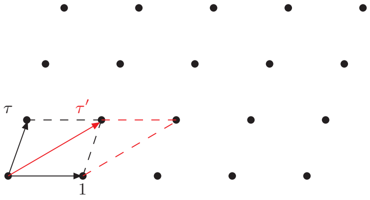
and require $\mathrm{Im}(\tau) > 0$.

Definition (The complex upper half-plane)

$$\mathbb{H} = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$$

Modular transformations

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).



Change of basis:
$$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

In terms of τ and τ':
$$\tau' = \frac{a\tau + b}{c\tau + d}$$

Modular forms

A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a **modular form** of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

1. f transforms under modular transformations as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for } \gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathrm{SL}_2(\mathbb{Z})$$

2. f is holomorphic on $\mathbb{H}$,
3. f is holomorphic at $i\infty$.

Define the $|_k\gamma$ operator by

$$(f|_k\gamma)(\tau) \quad = \quad (c\tau+d)^{-k} \cdot f(\gamma(\tau))$$

Congruence subgroups

Apart from $\mathrm{SL}_2(\mathbb{Z})$ we may also look at congruence **subgroups**, for example

$$\Gamma_0(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N, \; c \equiv 0 \bmod N \right\}$$

$$\Gamma(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N, \; b, c \equiv 0 \bmod N \right\}$$

Modular forms for congruence subgroups: Require “**nice**” transformation properties only for subgroup Γ (plus holomorphicity on $\mathbb{H}$ and at the cusps).

Modular forms

For a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ denote by $\mathcal{M}_k(\Gamma)$ the **space of modular forms of weight k**.

We have the inclusions

$$\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathcal{M}_k(\Gamma_0(N)) \subseteq \mathcal{M}_k(\Gamma_1(N)) \subseteq \mathcal{M}_k(\Gamma(N))$$

For $f \in \mathcal{M}_k(\Gamma(N))$:

$$\begin{aligned} f|_k\gamma &= f, & \gamma &\in \Gamma(N) \\ f|_k\gamma &\in \mathcal{M}_k(\Gamma(N)), & \gamma &\in \mathrm{SL}_2(\mathbb{Z})\backslash\Gamma(N) \end{aligned}$$

Notation

For $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ set

$$\bar{q} = \exp(2\pi i \tau), \qquad \bar{w} = \exp(2\pi i z)$$

Maps the complex **upper half-plane** $\tau \in \mathbb{H}$ **to** the **unit disk** $|\bar{q}| < 1$.

Trivialises periodicity with period 1:

$$\bar{q}(\tau + 1) = \bar{q}(\tau), \qquad \bar{w}(z+1) = \bar{w}(z)$$

Shifts with τ correspond to multiplication with $\bar{q}$:

$$\bar{q}(\tau + \tau) = \bar{q}(\tau) \cdot \bar{q}(\tau), \qquad \bar{w}(z+\tau) = \bar{w}(z) \cdot \bar{q}(\tau)$$

Iterated integrals of modular forms

Let $f_1, \dots, f_n$ be modular forms.

$$I(f_1, f_2, ..., f_n; q) = (2\pi i)^n \int\limits_{\tau_0}^{\tau} d\tau_1 f_1(\tau_1) \int\limits_{\tau_0}^{\tau_1} d\tau_2 f_2(\tau_2) \dots \int\limits_{\tau_0}^{\tau_{n-1}} d\tau_n f_n(\tau_n)$$

As basepoint we usually take $\tau_0 = i\infty$.

An integral over a modular form is in general **not** a modular form.

Analogy: An integral over a rational function is in general not a rational function.

Simple poles at $\tau = i\infty$

A modular form $f_k(\tau)$ is by definition holomorphic at the cusp and has a $\bar{q}$-expansion

$$f_k(\tau) \quad = \quad a_0 + a_1 \bar{q} + a_2 \bar{q}^2 + ..., \qquad\qquad \bar{q} = \exp(2\pi i \tau)$$

The transformation $\bar{q} = \exp(2\pi i \tau)$ transforms the point $\tau = i\infty$ to $\bar{q} = 0$ and we have

$$2\pi i \; f_k(\tau) d\tau \quad = \quad \frac{d\bar{q}}{\bar{q}} \left(a_0 + a_1 \bar{q} + a_2 \bar{q}^2 + ... \right).$$

Thus a modular form non-vanishing at the cusp $\tau = i\infty$ has a **simple pole** at $\bar{q} = 0$.

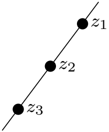
Subsection 2

Moduli spaces

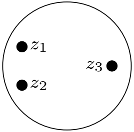
Moduli spaces

$\mathcal{M}_{g,n}$: Space of **isomorphism classes of** smooth (complex, algebraic) **curves of genus g with n marked points**.

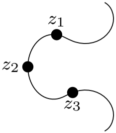
complex curve



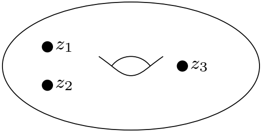
$\Leftrightarrow$



real surface



$\Leftrightarrow$



Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n - 3.$
Sphere has a unique shape
Use Möbius transformation to fix $z_{n-2} = 1$, $z_{n-1} = \infty$, $z_n = 0$
Coordinates are $(\mathbf{z_1}, \dots, \mathbf{z_{n-3}})$

Genus 1: $\dim \mathcal{M}_{1,n} = n.$
One coordinate describes the shape of the torus
Use translation to fix $z_n = 0$
Coordinates are $(\tau, \mathbf{z_1}, \dots, \mathbf{z_{n-1}})$

Iterated integrals

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \rightarrow M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \;=\; \gamma^* \omega_j.$$

The iterated integral is defined by

$$I_\gamma(\omega_1,...,\omega_k;\lambda) \;=\; \int\limits_0^\lambda d\lambda_1 f_1(\lambda_1) \int\limits_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \dots \int\limits_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Iterated integrals on $\mathcal{M}_{0,n}$

We are interested in differential one-forms, which have only simple poles:

$$\omega^{\mathrm{mpl}}(z_j) = \frac{dy}{y - z_j}.$$

Multiple polylogarithms:

$$G(z_1, \dots, z_k; y) = \int\limits_0^y \frac{dy_1}{y_1 - z_1} \int\limits_0^{y_1} \frac{dy_2}{y_2 - z_2} \dots \int\limits_0^{y_{k-1}} \frac{dy_k}{y_k - z_k}, \quad z_k \neq 0$$

Iterated integrals on $\mathcal{M}_{1,n}$

- Coordinates are $(\tau, z_1, ..., z_{n-1})$
- Decompose an arbitrary path along $d\tau$ and dz_j
- Two classes of iterated integrals:
  1. Integration along z
  2. Integration along τ
- What are the differential one-forms we want to integrate?

The Kronecker function

The **first Jacobi theta function** $\theta_1(z,q)$:

$$\theta_1(z,q) \;=\; -i \sum_{n=-\infty}^{\infty} (-1)^n q^{\left(n+\frac{1}{2}\right)^2} e^{i(2n+1)z}, \qquad q = e^{i\pi\tau}$$

The **Kronecker function** $F(z,\alpha,\tau)$:

$$F(z,\alpha,\tau) \;=\; \pi\theta_1'(0,q)\frac{\theta_1(\pi(z+\alpha),q)}{\theta_1(\pi z,q)\,\theta_1(\pi\alpha,q)} = \frac{1}{\alpha}\sum_{k=0}^{\infty} \mathbf{g}^{(\mathbf{k})}(\mathbf{z},\tau)\alpha^k$$

We are mainly interested in the coefficients $g^{(k)}(z,\tau)$ of the Kronecker function.

The coefficients $g^{(k)}(z, \tau)$ of the Kronecker function

Properties of $g^{(k)}(z, \tau)$:

1. **only simple poles** as a function of z
2. **quasi-periodic** as a function of z: Periodic by 1, quasi-periodic by τ.

$$
\begin{aligned}
g^{(k)}(z+1, \tau) &= g^{(k)}(z, \tau), \\
g^{(k)}(z+\tau, \tau) &= \sum_{j=0}^{k} \frac{(-2\pi i)^j}{j!} g^{(k-j)}(z, \tau)
\end{aligned}
$$

3. **almost modular**:

$$
g^{(k)}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \sum_{j=0}^{k} \frac{(2\pi i)^j}{j!} \left(\frac{cz}{c\tau+d}\right)^j g^{(k-j)}(z, \tau)
$$

Differential one-forms on $\mathcal{M}_{1,n}$

- To keep the discussion simple, we start with $\mathcal{M}_{1,2}$ with coordinates (τ, z):
  - One-forms from modular forms:

$$\omega_k^{\text{modular}} = 2\pi i \, f_k(\tau) d\tau$$

  - One-forms from the Kronecker function:

$$\omega_k^{\text{Kronecker}} = (2\pi i)^{2-k} \left[g^{(k-1)}(z - c_j, \tau)\, dz + (k-1)\, g^{(k)}(z - c_j, \tau) \frac{d\tau}{2\pi i} \right]$$

    with c_j being a constant.
- We allow the substitution $\tau \to K\tau$ with $K \in \mathbb{N}$.
- On $\mathcal{M}_{1,n}$ with coordinates $(\tau, z_1, ..., z_{n-1})$ we consider $z \to z_j$ with $1 \leq j \leq (n-1)$.

Iterated integrals on $\mathcal{M}_{1,n}$: Integration along z

Differential one-forms:

$$\omega_k^{\text{Kronecker},z}(z_j,\tau) = (2\pi i)^{2-k} g^{(k-1)}(z-z_j,\tau)\, dz$$

Elliptic multiple polylogarithms:

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_r \\ z_1 & \dots & z_r \end{smallmatrix} ; z; \tau \right) = (2\pi i)^{n_1+\dots+n_r-r} I\left(\omega_{n_1+1}^{\text{Kronecker},z}(z_1,\tau), \dots, \omega_{n_r+1}^{\text{Kronecker},z}(z_r,\tau); z \right)$$

- $\tau = \text{const}$
- meromorphic version, only simple poles
- not double periodic!

Iterated integrals on $\mathcal{M}_{1,n}$: Integration along τ

Differential one-forms:

$$\begin{aligned}\omega_k^{\mathrm{Kronecker},\tau}(z_j) &= (2\pi i)^{2-k}\,(k-1)\,g^{(k)}(z_j,\tau)\,\frac{d\tau}{2\pi i} \\ &= \frac{(k-1)}{(2\pi i)^k}\,g^{(k)}(z_j,\tau)\,\frac{d\bar{q}}{\bar{q}}\end{aligned}$$

- Integrate in $\bar{q}$
- No poles in $0 < |\bar{q}| < 1$.
- Possibly a simple pole at $\bar{q} = 0$ (“trailing zero”)

Subsection 3

Physics

The equal-mass sunrise

It is **not possible** to obtain an ε-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.
However by factoring off the (**non-algebraic**) expression ψ_1/π from the master integrals in the sunrise sector one obtains an ε-form:

$$I_1 = 4\varepsilon^2 I_{110}\left(2-2\varepsilon,x\right) \quad I_2 = -\varepsilon^2 \frac{\pi}{\psi_1} I_{111}\left(2-2\varepsilon,x\right) \quad I_3 = \frac{1}{\varepsilon}\frac{1}{2\pi i}\frac{d}{d\tau} I_2 + \frac{1}{24}\left(3x^2-10x-9\right)\frac{\psi_1^2}{\pi^2} I_2$$

If in addition one makes a (**non-algebraic**) change of variables from x to τ, one obtains

$$\frac{d}{d\tau} I \quad = \quad \varepsilon \, A(\tau) \, I,$$

where $A(\tau)$ is an ε-independent 3×3-matrix whose **entries are modular forms**.

The unequal-mass sunrise

After a redefinition of the basis of master integrals and a change of coordiantes from $(x, y_1, y_2) = (p^2/m_3^2, m_1^2/m_3^2, m_2^2/m_3^2)$ to (τ, z_1, z_2) one finds

$$\mathbf{A} \;=\; \varepsilon \sum_{j=1}^{N_L} C_j \, \omega_j, \qquad \text{with } \omega_j \text{ only simple poles},$$

where ω_j is either

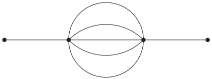
$$2\pi i \, f_k(\tau) \, d\tau,$$

where $f_k(\tau)$ is a modular form, or of the form

$$\omega_k(z_i, K\tau) \;=\; (2\pi i)^{2-k} \left[g^{(k-1)}(z_i, K\tau) \, dz_i + K(k-1) \, g^{(k)}(z_i, K\tau) \frac{d\tau}{2\pi i} \right]$$

Generalisations

- We understand by now very well Feynman integrals related to algebraic curves of genus 0 and 1. These correspond to iterated integrals on the moduli spaces $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$.
- The obvious generalisation is the generalisation to algebraic curves of higher genus g, i.e. iterated integrals on the moduli spaces $\mathcal{M}_{g,n}$.
- However, we also need the generalisation from curves to surfaces and higher dimensional objects: The geometry of the banana graphs with equal non-vanishing internal masses



are Calabi-Yau manifolds.

Section 2

General theorems on Feynman integrals

Motivation

- With Calabi-Yau manifolds appearing in the banana graphs it seems that we widely opened the door to the full complexity of algebraic geometry.
- We would like to have some general theorems, giving us some “upper bounds”: No Feynman integral is worse than ...

Two theorems

Theorem (Feynman integrals and $\mathcal{A}$-hypergeometric functions)

Any Feynman integral is a special case of a $\mathcal{A}$-hypergeometric function.

Theorem (Feynman integrals and periods)

Assume that all kinematic variables $x_1, \dots, x_{N_B}$ are rational and in the Euclidean region and $\nu_j \in \mathbb{Z}$. Then each coefficient $I^{(j)}_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(x_1, \dots, x_{N_B})$ in the Laurent expansion in ε is a numerical period.

Subsection 1

$\mathcal{A}$-hypergeometric functions

GKZ systems

- Gel'fand, Kapranov and Zelevinsky (GKZ) studied a system of partial differential equations, called GKZ system and its solution.
- The solutions are called **$\mathcal{A}$-hypergeometric functions** or GKZ hypergeometric functions.
- $\mathcal{A}$-hypergeometric functions are generalisations of the hypergeometric function ${}_pF_q$.

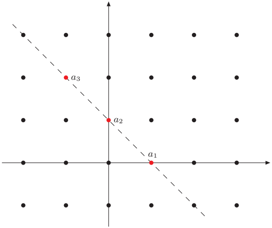
GKZ systems

A **$\mathcal{A}$-hypergeometric system** or **GKZ hypergeometric system** is defined by a vector $c \in \mathbb{C}^{k+1}$ and an n-element subset

$$\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{Z}^{k+1},$$

which satisfies the two conditions

1. $\mathcal{A}$ generates $\mathbb{Z}^{k+1}$ as an abelian group.
2. $a_1, \dots, a_n$ lie in a hyperplane.



GKZ systems

We denote by $\mathbb{L} \subset \mathbb{Z}^n$ the lattice of relations in $\mathcal{A}$:

$$\mathbb{L} \quad = \quad \{ (l_1, \dots, l_n) \in \mathbb{Z}^n \mid l_1 a_1 + \dots + l_n a_n = 0 \}.$$

The **GKZ hypergeometric system** associated with $\mathcal{A}$ and c is the following system of differential equations for a function ϕ of n variables $x_1, \dots, x_n$:

- For every $(l_1, \dots, l_n) \in \mathbb{L}$ one differential equation

$$\left[\prod_{l_j > 0} \left(\frac{\partial}{\partial x_j} \right)^{l_j} - \prod_{l_j < 0} \left(\frac{\partial}{\partial x_j} \right)^{-l_j} \right] \phi \quad = \quad 0$$

- $(k+1)$ differential equations

$$\left[\sum_{j=1}^{n} a_j x_j \frac{\partial}{\partial x_j} - c \right] \phi \quad = \quad 0$$

$\phi(x_1, \dots, x_n)$ is called a **$\mathcal{A}$-hypergeometric function**.

Euler-Mellin integrals

Euler-Mellin integrals are $\mathcal{A}$-hypergeometric functions

$$\int\limits_{\mathcal{C}} d^k z \left(\prod_{j=1}^{k} z_j^{\nu_j - 1} \right) \left[G(z,x) \right]^{-\nu_0}, \qquad G(z,x) = \sum_{j=1}^{n} x_j z_1^{a_{1j}} \dots z_k^{a_{kj}}$$

with

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_{11} & a_{12} & \dots & a_{1n} \\ \dots & & & \dots \\ a_{k1} & a_{k2} & \dots & a_{kn} \end{pmatrix}, \quad c = \begin{pmatrix} -\nu_0 \\ -\nu_1 \\ \dots \\ -\nu_k \end{pmatrix}.$$

Note: **One variable x_j for every monomial in G.**

Lee-Pomeransky representation

Lee-Pomeransky representation:

$$I = C \int\limits_{\mathcal{C}} d^{n_{\mathrm{int}}} z \left(\prod_{j=1}^{n_{\mathrm{int}}} z_j^{\nu_j - 1} \right) \left[\mathcal{G}(z,x) \right]^{-\frac{D}{2}},$$
$$\textit{with} \;\; \mathcal{G}(z,x) = \mathcal{U}(z) + \mathcal{F}(z,x).$$

- In general the number of kinematic variables N_B **does not equal** the number of monomials in $\mathcal{G}(z,x)$.
- Consider a **generalised Lee-Pomeransky polynomial** $G(z,x)$ with one variable x_j for every monomial in G.
- Recover $\mathcal{G}$ from G as the case, where the additional variables take special values (for example the value 1).

The double box:

The original Lee-Pomeransky polynomial with one variable x:

$$\begin{aligned} \mathcal{G} \; = \; & z_1 z_5 + z_1 z_6 + z_1 z_7 + z_2 z_5 + z_2 z_6 + z_2 z_7 + z_3 z_5 + z_3 z_6 + z_3 z_7 + z_1 z_4 + z_2 z_4 \\ & + z_3 z_4 + z_4 z_5 + z_4 z_6 + z_4 z_7 + x z_2 z_3 z_4 + x z_2 z_3 z_5 + x z_2 z_3 z_6 + x z_2 z_3 z_7 + x z_5 z_6 z_1 \\ & + x z_5 z_6 z_2 + x z_5 z_6 z_3 + x z_5 z_6 z_4 + x z_2 z_4 z_6 + x z_3 z_4 z_5 + z_1 z_4 z_7 \end{aligned}$$

The generalised Lee-Pomeransky polynomial G with variables $x_1, \dots, x_{26}$:

$$\begin{aligned} \mathcal{G} \; = \; & x_1 z_1 z_5 + x_2 z_1 z_6 + x_3 z_1 z_7 + x_4 z_2 z_5 + x_5 z_2 z_6 + x_6 z_2 z_7 + x_7 z_3 z_5 + x_8 z_3 z_6 + x_9 z_3 z_7 \\ & + x_{10} z_1 z_4 + x_{11} z_2 z_4 + x_{12} z_3 z_4 + x_{13} z_4 z_5 + x_{14} z_4 z_6 + x_{15} z_4 z_7 + x_{16} z_2 z_3 z_4 \\ & + x_{17} z_2 z_3 z_5 + x_{18} z_2 z_3 z_6 + x_{19} z_2 z_3 z_7 + x_{20} z_5 z_6 z_1 + x_{21} z_5 z_6 z_2 + x_{22} z_5 z_6 z_3 \\ & + x_{23} z_5 z_6 z_4 + x_{24} z_2 z_4 z_6 + x_{25} z_3 z_4 z_5 + x_{26} z_1 z_4 z_7 \end{aligned}$$

Feynman integrals and $\mathcal{A}$-hypergeometric functions

Theorem (Feynman integrals and $\mathcal{A}$-hypergeometric functions)

Any Feynman integral $I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B})$ is a special case of a $\mathcal{A}$-hypergeometric function in more variables $x_1, \dots, x_n$, where the additional variables take special values.

Subsection 2

Periods

Periods as integrals over algebraic functions

The periods of $\exp(z)$ and $\wp(z)$ can be expressed as **integrals involving only algebraic functions**.

- Period of the exponential function:

$$2\pi i = 2i \int\limits_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}.$$

- Periods of Weierstrass's $\wp$-function: Assume that g_2 and g_3 are two given algebraic numbers. Then

$$\omega_1 = 2 \int\limits_{t_1}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad \omega_2 = 2 \int\limits_{t_3}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}},$$

where t_1, t_2 and t_3 are the roots of the cubic equation $4t^3 - g_2 t - g_3 = 0$.

Numerical periods

Kontsevich and Zagier suggested the following generalisation:

Definition

A **numerical period** is a **complex number** whose real and imaginary parts are values of **absolutely convergent integrals** of **rational functions** with **rational coefficients**, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Remarks:

- One can replace "rational" with "algebraic".
- The set of all periods is countable.

Examples

- π is a numerical period:

$$\pi = \iint\limits_{x^2+y^2 \leq 1} dx\, dy.$$

- $\ln(2)$ is a numerical period:

$$\ln(2) = \int\limits_1^2 \frac{dx}{x}.$$

Feynman integrals and periods

A Feynman integral has a **Laurent expansion** in ε:

$$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B}) = \sum_{j=j_{\mathrm{min}}}^{\infty} \varepsilon^j \, I^{(j)}_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(x_1, \dots, x_{N_B})$$

Theorem (Feynman integrals and periods)

Assume that all kinematic variables $x_1, \dots, x_{N_B}$ are rational and in the Euclidean region and $\nu_j \in \mathbb{Z}$. Then $I^{(j)}_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(x_1, \dots, x_{N_B})$ is a numerical period.

- Proof based on the resolution of singularities.
- Practical application: Sector decomposition.

Section 3

Conclusions

Summary

- Feynman integrals are needed for precision calculations in perturbative quantum field theory.
- Various integral representations.
- Method of differential equations is a powerfull tool for computing Feynman integrals.
- Deep connection with mathematics:
  - Intersection numbers
  - Elliptic curves
  - $\mathcal{A}$-hypergeometric functions