Exercises for the course “Feynman integrals”

Sheet 4

Exercise 6

Let $\vec{I} = (I_{\nu_1}, \dots, I_{\nu_{N_{\text{master}}}})^T$ be a set of master integrals, which satisfies the differential equation

$$(d + A)\vec{I} \;=\; 0.$$

Let $\vec{I}'$ be another set of master integrals, related to the first one by an invertible $N_{\text{master}} \times N_{\text{master}}$-matrix $U(\varepsilon, x)$:

$$\vec{I}' \;=\; U\vec{I}.$$

Show that $\vec{I}'$satisfies the differential equation

$$\left(d + A'\right)\vec{I}' \;=\; 0$$

with

$$A' \;=\; UAU^{-1} + UdU^{-1}.$$

Solution:

We have

$$\vec{I} \;=\; U^{-1}\vec{I}'$$

and

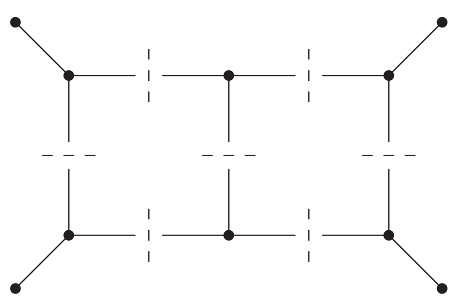
$$d\vec{I} \;=\; d\left(U^{-1}\vec{I}'\right) \;=\; U^{-1}d\vec{I}' + \left(dU^{-1}\right)\vec{I}'.$$

Thus

$$d\vec{I}' \;=\; Ud\vec{I} - \left(UdU^{-1}\right)\vec{I}' \;=\; -UA\vec{I} - \left(UdU^{-1}\right)\vec{I}' \;=\; -\left(UAU^{-1} + UdU^{-1}\right)\vec{I}'.$$

Exercise 7

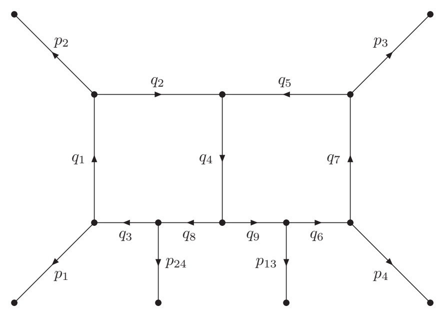
Work out a one-dimensional integral representation for the maximal cut of the double box integral $I_{111111100}$ (kinematics and definition of $\tilde{G}$ as in exercise 5)



To work out the maximal cut it is simpler to use a loop-by-loop approach: First write down a Baikov representation for a one-loop sub-graph, and then a second Baikov representation for the second loop.

Solution:

We label the internal edges as in the following figure:



We first consider the loop with edges e_1, e_2, e_3 and e_4 and then the second loop with edges e_5, e_6, e_7 and an edge with momentum $k_2 + p_1$. The latter edge is introduced by integrating out the first loop with loop momentum k_1. The loop-by-loop approach has the advantage that we only need eight Baikov variables z_1-z_7 and z_9, the variable z_8 is absent. The Baikov representation reads

$$I_{111111100} = \frac{e^{2\varepsilon\gamma_{\mathrm{E}}}\left(\mu^2\right)^{7-D}}{64\pi^3\Gamma\left(\frac{D-3}{2}\right)^2}\left[G\left(p_1,p_2,p_3\right)\right]^{\frac{4-D}{2}} \int\limits_{C} d^8z \left[G\left(p_1,p_2,k_2\right)\right]^{\frac{4-D}{2}}\left[G\left(k_1,p_1,p_2,k_2\right)\right]^{\frac{D-5}{2}}\left[G\left(k_2,p_1,p_2,p_3\right)\right]^{\frac{D-5}{2}} \frac{1}{z_1z_2z_3z_4z_5z_6z_7}.$$

We have

$$\begin{aligned}
G\left(p_1,p_2,p_3\right) &= \frac{1}{4}st\left(s+t\right), \\
G\left(p_1,p_2,k_2\right)\big|_{z_1=z_2=z_3=z_4=z_5=z_6=z_7=0} &= \frac{1}{4}s\left(s+t-z_9\right)\left(t-z_9\right), \\
G\left(k_1,p_1,p_2,k_2\right)\big|_{z_1=z_2=z_3=z_4=z_5=z_6=z_7=0} &= \frac{1}{16}s^2\left(t-z_9\right)^2, \\
G\left(k_2,p_1,p_2,p_3\right)\big|_{z_1=z_2=z_3=z_4=z_5=z_6=z_7=0} &= \frac{1}{16}s^2z_9^2.
\end{aligned}$$

Thus

$$\mathrm{MaxCut}\, I_{111111100} = \left(2\pi i\right)^7 \frac{e^{2\varepsilon\gamma_{\mathrm{E}}}\left(\mu^2\right)^{7-D}2^{6-2D}}{\pi^3\Gamma\left(\frac{D-3}{2}\right)^2} s^{D-6}t^{\frac{4-D}{2}}\left(s+t\right)^{\frac{4-D}{2}} \int\limits_{C'} dz_9\, z_9^{D-5}\left(t-z_9\right)^{\frac{D-6}{2}}\left(s+t-z_9\right)^{\frac{4-D}{2}}.$$

The integration region $\mathcal{C}'$ is obtained from the conditions

$$\left.\frac{G(k_2,p_1,p_2,p_3)}{G(p_1,p_2,p_3)}\right|_{z_1=z_2=z_3=z_4=z_5=z_6=z_7=0} = \frac{1}{4}\frac{sz_9^2}{t(s+t)} > 0,$$
$$\left.\frac{G(k_1,p_1,p_2,k_2)}{G(p_1,p_2,k_2)}\right|_{z_1=z_2=z_3=z_4=z_5=z_6=z_7=0} = \frac{1}{4}\frac{s(t-z_9)}{(s+t-z_9)} > 0.$$

The exercise was about deriving this integral representation.

The integration can be performed as follows: Let's assume $t>0$ and $s<-t$. Then

$$\mathcal{C}' = \left]-\infty, s+t\right] \cup \left[t,\infty\right[,$$

and

$$\int\limits_{-\infty}^{s+t} dz_9\, z_9^{D-5}\,(t-z_9)^{\frac{D-6}{2}}\,(s+t-z_9)^{\frac{4-D}{2}} =$$
$$\frac{\Gamma\left(\frac{6-D}{2}\right)\Gamma(5-D)}{\Gamma\left(\frac{16-3D}{2}\right)}(s+t)^{D-5}\,{}_2F_1\left(5-D,\frac{6-D}{2},\frac{16-3D}{2};\frac{t}{s+t}\right),$$
$$\int\limits_{t}^{\infty} dz_9\, z_9^{D-5}\,(t-z_9)^{\frac{D-6}{2}}\,(s+t-z_9)^{\frac{4-D}{2}} =$$
$$-\frac{\Gamma\left(\frac{D-4}{2}\right)\Gamma\left(\frac{5-D}{2}\right)}{\sqrt{\pi}}2^{4-D}t^{D-5}\,{}_2F_1\left(5-D,\frac{D-4}{2},\frac{6-D}{2};\frac{s+t}{t}\right).$$

Combining the results we obtain

$$\mathrm{MaxCut}\, I_{111111100}(4-2\varepsilon) = (2\pi i)^7\,\frac{\left(\mu^2\right)^3}{4\pi^4 s^2 t\varepsilon} + O\left(\varepsilon^0\right).$$