Exercises for the course "Feynman integrals"
Sheet 5

Exercise 8

Consider the elliptic curve defined by the quartic polynomial $P(w,z) = 0$, where

$$P(w,z) \;=\; w^2 - z(z+4)\left[z^2 + 2(1+x)z + (1-x)^2\right].$$

x is a parameter. Determine two independent periods of the elliptic curve.

Solution:

Let's first consider the case of an elliptic curve defined by a factorised quartic polynomial

$$P(w,z) \;=\; w^2 - (z-z_1)(z-z_2)(z-z_3)(z-z_4).$$

$z_1, \dots, z_4$ are the roots. We may assume that the roots are distinct. The integral

$$\int\limits_{z_i}^{z_j} \frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}}$$

between any pair of roots is a half-period. In order to get two independent periods, we may for example choose the pairs (z_2, z_3) and (z_4, z_3). Thus we obtain

$$\begin{aligned}
\psi_1 &= 2\int\limits_{z_2}^{z_3} \frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}} \\
&= \frac{4}{\sqrt{(z_3-z_1)(z_4-z_2)}} K\left(\sqrt{\frac{(z_3-z_2)(z_4-z_1)}{(z_3-z_1)(z_4-z_2)}}\right), \\
\psi_2 &= 2\int\limits_{z_4}^{z_3} \frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}} \\
&= \frac{4i}{\sqrt{(z_3-z_1)(z_4-z_2)}} K\left(\sqrt{\frac{(z_2-z_1)(z_4-z_3)}{(z_3-z_1)(z_4-z_2)}}\right).
\end{aligned}$$

$K(x)$ denotes the complete elliptic integral of the first kind:

$$K(x) \;=\; \int\limits_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}.$$

For the case at hand, the roots of the quartic polynomial are

$$z_1 = -4, \quad z_2 = -\left(1+\sqrt{x}\right)^2, \quad z_3 = -\left(1-\sqrt{x}\right)^2, \quad z_4 = 0.$$

We introduce the modulus k and the complementary modulus k' through

$$k^2 = \frac{(z_3-z_2)(z_4-z_1)}{(z_3-z_1)(z_4-z_2)}, \qquad k'^2 = \frac{(z_2-z_1)(z_4-z_3)}{(z_3-z_1)(z_4-z_2)}.$$

Explicitly we have

$$k^2 = \frac{16\sqrt{x}}{(1+\sqrt{x})^3(3-\sqrt{x})}, \qquad k'^2 = \frac{(1-\sqrt{x})^3(3+\sqrt{x})}{(1+\sqrt{x})^3(3-\sqrt{x})}.$$

This gives

$$\begin{aligned} \psi_1 &= \frac{4}{(1+\sqrt{x})^{\frac{3}{2}}(3-\sqrt{x})^{\frac{1}{2}}} K(k), \\ \psi_2 &= \frac{4i}{(1+\sqrt{x})^{\frac{3}{2}}(3-\sqrt{x})^{\frac{1}{2}}} K(k'). \end{aligned}$$

Exercise 9

At the end of the tutorial let's play a game: **Hironaka's polyhedra game**. This game is played by two players, A and B. They are given a finite set M of points $m = (m_1, ..., m_n)$ in $\mathbb{N}_+^n$, the first quadrant of $\mathbb{N}^n$. We denote by $\Delta \subset \mathbb{R}_+^n$ the positive convex hull of the set M. It is given by the convex hull of the set

$$\bigcup_{m\in M} \left(m+\mathbb{R}_+^n\right).$$

The two players compete in the following game:

1. Player A chooses a non-empty subset $S \subseteq \{1, ..., n\}$.
2. Player B chooses one element i out of this subset S.

Then, according to the choices of the players, the components of all $(m_1, ..., m_n) \in M$ are replaced by new points $(m_1', ..., m_n')$, given by:

$$\begin{aligned} m_j' &= m_j, \quad \text{if } j \neq i, \\ m_i' &= \sum_{j\in S} m_j - 1. \end{aligned}$$

This defines the set M'. One then sets $M = M'$ and goes back to step 1. Player A wins the game if, after a finite number of moves, the polyhedron Δ is of the form

$$\Delta = m + \mathbb{R}_+^n,$$

i.e. generated by one point. If this never occurs, player B has won.

You may play the game with a fellow student. Actually, one may show that player A can always win the game and you may want to think about a winning strategy for player A (this is not easy).

Solution:

The main purpose of this exercise is to see how the points m move in $\mathbb{N}_+^n$ after each round. Giving a winning strategy for player A consists in defining for each configuration a tuple with an ordering criteria like lexicographical ordering, such that $(0,0,\dots)$ correspond to a polyhedron generated by one point and each move of player A will decrease the tuple, independently of the choice of player B. Examples of possible winning strategies are a little bit technical, examples can be found in arXiv:0709.4092.

A winning strategy corresponds in mathematics to an algorithm for the resolution of singularities through a sequence of blow-ups.

A winning strategy corresponds in physics to a terminating algorithm for the computation of a Feynman integral through sector decomposition.