On the twistor origin of celestial W-infinity symmetry.

Lionel Mason

The Mathematical Institute, Oxford
lmason@maths.ox.ac.uk

Celestial Sphere: holography, CFT and amplitudes
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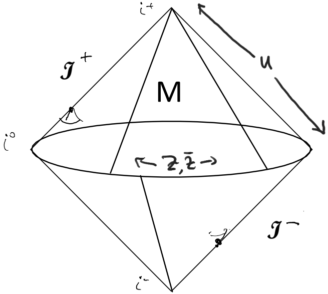
Work with: Tim Adamo & Atul Sharma 2103.16984, 21??...

Revisits: M & Wolf CMP, 288, '09, CMP, and M & Skinner CMP 294, '10 in light of recent developments.

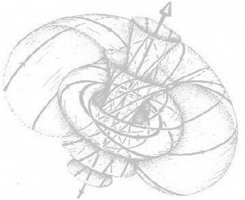
Slogan: SD twistor sigma model or ambitwistor string OPE corresponds to celestial W_∞ OPE from twistors at $\mathscr{I}$.

Twistors at null infinity, and amplitudes

- CCFT studies massless amplitudes via OPES of $\mathscr{I}$ data.
- Newman's good cuts attempt to rebuild space-time from $\mathscr{I}$ data.
- Yields instead '$\mathcal{H}$-space' a complex self-dual space-time.



- Penrose's nonlinear graviton arose from Newman's $\mathcal{H}$-space.
- Twistor's non-locality $\rightsquigarrow$ extend out to $\mathscr{I}^{\pm}$.
- Asymptotic Twistor space $\mathbb{PT} = \mathbb{CP}^3$ is 3d space describing 4d physics.



Outline

1. Lw_∞, Poisson diffeos and twistors for self-dual gravity.
2. Asymptotic twistor spaces from the asymptotic shear.
3. Twistor sigma model for gravity amplitudes.
4. $Lw_{1+\infty}$-symmetry as soft expansion of gravity vertex operators.

W-infinity from Poisson diffeos of plane

W_N = higher spin symmetries in 2d CFT.
When $N \to \infty$ classical limit w_∞ = Poisson diffeos of plane.

- Plane has coords $\mu^{\dot\alpha}$, $\dot\alpha = \dot 0, \dot 1$ with Poisson bracket

$$\{f, g\} := \varepsilon^{\dot\alpha\dot\beta} \frac{\partial f}{\partial \mu^{\dot\alpha}} \frac{\partial g}{\partial \mu^{\dot\alpha}} , \qquad \varepsilon^{\dot\alpha\dot\beta} = \varepsilon^{[\dot\alpha\dot\beta]}, \quad \varepsilon^{\dot 0 \dot 1} = 1 .$$

- Basis of $w_{1+\infty} \leftrightarrow$ polynomial hamiltonians

$$w^p_m = (\mu^{\dot 0})^{p-m-1}(\mu^{\dot 1})^{p+m-1} , \qquad |m| \leq p-1, \quad 2p-2 \in \mathbb{N}$$

- Poisson brackets $\leftrightarrow$ commutation relations of $w_{1+\infty}$.

$$\{w^p_m, w^q_n\} = (2(p-1)n - 2(q-1)m) w^{p+q-2}_{m+n} .$$

- Loop algebra $Lw_{1+\infty}$, loop coord $z = \mathrm{e}^{i\theta}$, and generators

$$g^p_{m,r} = w^p_m / z^r , \qquad r \in \mathbb{Z}.$$

- Poisson brackets now

$$\{g^p_{m,r}, g^q_{n,s}\} = (2(p-1)n - 2(q-1)m)\, g^{p+q-2}_{m+n,r+s} .$$

Twistor construction for self-dual space-times

Flat twistor space: $\mathbb{T} = \mathbb{C}^4$, projectively $\mathbb{PT}$ with hgs coords:

$$W = (\lambda_\alpha, \mu^{\dot\alpha}) \in \mathbb{T}, \qquad W \sim aW\,, a \neq 0\,.$$

We have

- Projection $\mathbb{PT} - \mathbb{CP}^1_{\{\lambda_\alpha = 0\}}$ onto celestial sphere

$$p : \mathbb{PT} \to \mathbb{CP}^1\,, \qquad p(W) = \lambda_\alpha\,, \quad z = \lambda_1/\lambda_0\,.$$

- Poisson bracket:

$$\{f, g\} = \varepsilon^{\dot\alpha\dot\beta} \frac{\partial f}{\partial \mu^{\dot\alpha}} \frac{\partial g}{\partial \mu^{\dot\beta}} = \left[\frac{\partial f}{\partial \mu} \frac{\partial g}{\partial \mu}\right]\,.$$

Theorem (Penrose 1976)

There is a 1 : 1 *correspondence between:*

- *SD Ricci flat holomorphic metrics on regions in* $\mathbb{C}^4$, *and*
- *deformations* $P\mathcal{T}$ *of twistor space* $\mathbb{PT}$ *preserving fibration* $p : \mathcal{T} \to \mathbb{CP}^1$ *and Poisson structure on fibres of* p.

Patching functions for deformations

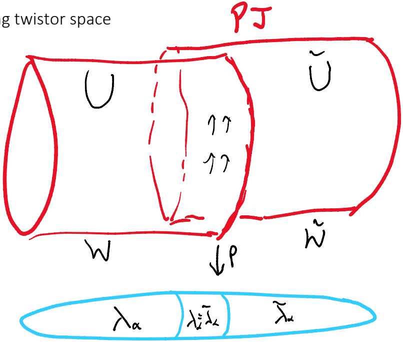
SD metrics $\leftrightarrow$ deformations $\mathbb{PT} \to \mathcal{PT} \leftrightarrow$ Patching functions

- Cover $\mathbb{PT} - \mathbb{CP}^1$ by patches

$$U = \{\lambda_0 \neq 0\}, \qquad \tilde{U} = \{\lambda_1 \neq 0\}$$

  coords $W = (\lambda_\alpha, \mu^{\dot\alpha})$ and $\tilde{W} = (\lambda_\alpha, \tilde{\mu}^{\dot\alpha})$ resp..
- 'Patch' $\tilde{W} = \tilde{W}(W)$ on $U \cap \tilde{U}$;
- $\lambda_\alpha = \tilde{\lambda}_\alpha$ to preserve $p : \mathcal{PT} \to \mathbb{CP}^1_{\lambda_\alpha}$.

Patching twistor space



Patching function $\tilde{\mu}^{\dot{\alpha}} = \tilde{\mu}^{\dot{\alpha}}(\lambda_\alpha, \mu^{\dot{\alpha}})$ must preserve $\{,\} \Rightarrow$

Proposition

Space of SD metrics $\simeq$ Loop group of Poisson diffeos.

- Coord transform preserves $\{,\} \leftrightarrow$ generating function: $G(\lambda_\alpha, \mu^{\dot{0}}, \tilde{\mu}^{\dot{1}})$, homogeneity degree 2 determines:

$$\mu^{\dot{1}} = \frac{\partial G}{\partial \mu^{\dot{0}}}\,, \qquad \tilde{\mu}^{\dot{0}} = \frac{\partial G}{\partial \tilde{\mu}^{\dot{1}}}\,.$$

- Infinitesimally $g = \delta G \in Lw_{1+\infty}$ can be expanded in

$$g^p_{m,r} = \frac{(\mu^{\dot{0}})^{p-m-1}(\mu^{\dot{1}})^{p+m-1}}{\lambda_0^{2p-4-r}\lambda_1^r}\,, \qquad p-2 \in \mathbb{N}, \quad r \in \mathbb{Z}\,.$$

Asymptotic Twistor space

Penrose's nonlinear graviton at $\mathscr{I}$ with deformed $\bar{\partial}$ operator.

Twistor space $\mathcal{T} = \mathbb{C}^4$ or projective $\mathbb{PT}$, homogeneous coords:

$$W = (\lambda_\alpha, \mu^{\dot\alpha}) \in \mathbb{T}, \qquad W \sim aW\,, a \neq 0\,.$$

Poisson bracket $\{f, g\} = \varepsilon^{\dot\alpha\dot\beta} \frac{\partial f}{\partial \mu^{\dot\alpha}} \frac{\partial g}{\partial \mu^{\dot\beta}} = \left[\frac{\partial f}{\partial \mu} \frac{\partial g}{\partial \mu}\right]$ as before.

- $q : \mathcal{T} \to \mathscr{I}$ given in Bondi coordinates

$$z = \frac{\lambda_1}{\lambda_0}\,, \qquad u = \frac{\mu^{\dot\alpha}\bar\lambda_{\dot\alpha}}{|\lambda_0|^2 + |\lambda_1|^2}\,.$$

- Asymptotic shear $C_{\bar z\bar z} d\bar z$ pulls back to $\mathcal{T}$ to define

$$\mathbf{h} = \int^u du' C_{\bar z\bar z}(u', z, \bar z) d\bar z = h(\mu^{\dot\alpha}\bar\lambda_{\dot\alpha}, \lambda, \bar\lambda)[\bar\lambda d\bar\lambda] \in \Omega^{0,1}(2)\,.$$

- $\bar\partial$-operator on $\mathcal{T}$ is deformed by 'Hamiltonian' h to

$$\bar\partial_h f := \bar\partial_0 f + \{h, f\}\,.$$

The self-dual space-time from holomorphic curves

To reconstruct self-dual space-time

$$(M^4, g) = \{ \text{ Holomorphic degree-1 } \mathbb{CP}^1\text{s in } P\mathcal{T}\}$$

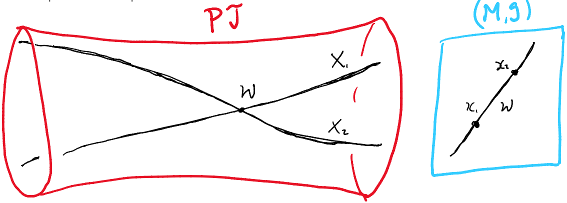
- Parametrize the $\mathbb{CP}^1_{x,\sigma} \subset \mathbb{P}\mathcal{T}$, with hgs coords (σ_0, σ_1) by

$$\lambda_\alpha = \left(\frac{1}{\sigma_0}, \frac{1}{\sigma_1}\right) = \frac{(1, z)}{\sigma_0}, \qquad \mu^{\dot\alpha} = \frac{x^{0\dot\alpha}}{\sigma_0} + \frac{x^{1\dot\alpha}}{\sigma_1} + M^{\dot\alpha},$$

- d-bar eq for $\mathbb{C}$-curves in deformed $\mathbb{P}\mathcal{T}$:

$$\bar\partial_\sigma \mu^{\dot\alpha} = \{\mu^{\dot\alpha}, h\} = \varepsilon^{\dot\alpha\dot\beta}\frac{\partial h}{\partial \mu^{\dot\beta}},$$

Correspondence with space-time



Sigma model action and MHV generating function

- Curve in $P\mathcal{T}$ parametrized by $\mu^{\dot\alpha} = x^{0\dot\alpha}/\sigma_0 + x^{1\dot\alpha}/\sigma_1 + M^{\dot\alpha}$,
- Follows from action

$$S[\mu^{\dot\alpha}, x] = \int D\sigma \left([M\bar\partial_\sigma M] + 2h(\lambda,\mu) \right)$$

Key proposition: [Adamo, M., Sharma, 2103.16984]
Given x, $\exists! \mu^{\dot\alpha}$ for which the on-shell action $S^{os}[x,h]$ for $x \in M$ and given h yields:

1. The Kahler (Plebanski) scalar $\Omega(x^{0\dot\alpha}, x^{1\dot\alpha})$ for g_+,
2. the generating function for MHV amplitudes

$$\mathcal{M}(1,2,h) = \langle 1\,2 \rangle^4 \int_M d^4x\, \mathrm{e}^{(k_1+k_2)\cdot x}\, S^{os}[x,h]\,.$$

MHV generating function, trees and Hodge formula

Starting from the MHV generating function

$$\mathcal{M}(1,2,h) = \langle 1\,2\rangle^4 \int_M d^4x\, \mathrm{e}^{(k_1+k_2)\cdot x}\, S^{os}[x,h]$$

- perturbatively expand in h in momentum eigenstates

$$h = \sum_{i=3}^{n} h_i\,, \qquad h_i = \int \frac{ds}{s^3}\bar{\delta}^2(s\lambda_\alpha - \kappa_{i\alpha})\,\mathrm{e}^{is[\mu\,\tilde{\kappa}_i]}\,.$$

- On-shell action has tree expansion (ignoring $O(h_i^2)$)

$$S_{\mathbb{PT}}[M,h] = \langle V_{h_3}\dots V_{h_n}\rangle_{\text{tree}}$$

with vertex operators $V_h = \int_{\mathbb{CP}^1} hD\sigma$ and propagators

$$\frac{[\partial_\mu h_i\, \partial_\mu h_j]}{\langle i\,j\rangle} = \frac{[i\,j]}{\langle i\,j\rangle} h_i h_j$$

- Yields tree-diagram formalism of Bern et. al. 1998.

Matrix-tree theorem gives $\langle V_{h_3}\dots V_{h_n}\rangle_{\text{tree}} = \det{}'\mathbb{H}$

$\rightsquigarrow$ Hodges reduced determinant formula, [cf Feng-He'12].

Sigma model at higher MHV degree

For N^{k-2}MHV need k ASD particles:

- ASD wave functions as momentum eigenstates

$$\tilde{h}_r(W_r) = \int s^5 ds\, \bar{\delta}^2(s\lambda - \kappa)\mathrm{e}^{is[\mu\kappa]} \in H^1(\mathcal{O}(-6))\,.$$

- Insert ASD particles at $W_r \in \mathbb{T}$ and $\sigma_r \in \mathbb{CP}^1$, $r = 1, \ldots, k$:

$$W(\sigma) = \sum_{r=1}^{k} \frac{W_r}{\sigma - \sigma_r} + (0, M^{\dot{\alpha}}) : \mathbb{CP}^1 \to \mathbb{PT}\,.$$

- There exists unique $W(\sigma)$ with M of weight $(-1, 0)$.
- Action is now simply

$$S[W(\sigma), W_r, \sigma_r, h] = \int_{\mathbb{CP}^1} d\sigma \left([M\,\bar{\partial} M] + 2h\right)$$

Propn: On-shell action $S^{os}[W_r, \sigma_r, h]$ generates N^{k-2}MHV tree-amplitudes

The formula on background h is:

$$\mathcal{M}(1^-,\ldots,k^-,h)=\int_{(\mathbb{CP}^1\times\mathbb{PT})^k} S^{os}[W_r,\sigma_r,h]\,\det{}'\tilde{\mathbb{H}}\prod_{r=1}^{k}\tilde{h}_r D^3W_r d\sigma_r\,.$$

here we have inserted $\det{}'\tilde{\mathbb{H}}$, for the 'conjugate' Hodge matrix

$$\tilde{\mathbb{H}}_{ij}=\begin{cases}\frac{\langle\lambda_r\lambda_s\rangle}{\sigma_r-\sigma_s} & r\neq s\\ -\sum_q\frac{\langle\lambda_r\lambda_q\rangle}{\sigma_r-\sigma_q}\,, & r=s\,.\end{cases}$$

Expanding $h=\sum_{i=k+1}^n h_i$ as before gives

$$\mathcal{M}=\int_{(\mathbb{CP}^1\times\mathbb{PT})^k}\langle h_{k+1}\ldots h_n\rangle_{\text{tree}}\det{}'\tilde{\mathbb{H}}\prod_{r=1}^{k}\tilde{h}_r D^3W_r d\sigma_r\,,$$

$$=\int_{(\mathbb{CP}^1)^n\times\mathbb{PT}^k}\det{}'\mathbb{H}\det{}'\tilde{\mathbb{H}}\prod_{r=1}^{k}\tilde{h}_r D^3W_r d\sigma_r$$

prove by reduction to Cachazo-Skinner formula.

Recall, $w_{1+\infty}$ = Poisson diffeos of $\mu^{\dot\alpha}$-plane.

- With $\mu^{\dot\alpha} = x^{0\dot\alpha}/\sigma_0 + x^{1\dot\alpha}/\sigma_1 + M^{\dot\alpha}$, sigma model action

$$S_{\mathbb{PT}}[\mu^{\dot\alpha}] = \int D\sigma \left([\mu\bar\partial_\sigma\mu] + 2h\right) + [\mu_{\sigma_0=0}\, x^0] + [\mu_{\sigma_1=0} x^1]\,.$$

- Gauge symmetry under Poisson diffeos $\leftrightarrow g(W, \bar\lambda)$ gives

$$\delta\mu^{\dot\alpha} = \{g, \mu^{\dot\alpha}\} = \varepsilon^{\dot\alpha\dot\beta}\frac{\partial g}{\partial\mu^{\dot\beta}}\,, \qquad \delta h = \bar\partial_0 g + \{h, g\} =: \bar\partial_h g\,.$$

- Symmetry $\Rightarrow \delta h = 0 \Rightarrow g$ holomorphic, $\bar\partial_h g = 0$.
- If g global, $h = 0$, $\rightsquigarrow g$ quadratic in $g = A_{IJ}W^I W^J$ gives Poincaré symmetry.
- If g is local, defines an infinitesimal generating function $\rightsquigarrow$ $Lw_{1+\infty}$ and graviton, i.e., perturbation of metric.

Worldsheet CFT for symmetries and gravitons

- OPE of $f(W), g(W)$ at $h = 0$ is

$$f(W(\sigma)) \cdot g(W(\sigma')) \sim \frac{\{f, g\}}{\sigma - \sigma'} + \dots .$$

- Čech-Dolbeault correspondence

$$\{g(W) \text{ holomorphic on } U \cap \tilde{U}\} \leftrightarrow \{h \in \Omega^{0,1}(2)/\bar{\partial}(*), \bar{\partial} h = 0\}.$$

so that: $\oint_\Sigma gD\sigma = \int_\Sigma hD\sigma$.

- Such a local g, gives Čech version of vertex operators

$$Q_g := \oint gD\sigma = \int hD\sigma = V_h \,.$$

Propn: Local symmetries = gravity vertex operators.

Soft expansion versus mode expansion for $w_{1+\infty}$

- Čech momentum eigenstate is
$$g_{\omega,\kappa,\tilde{\kappa}} = \frac{\langle\lambda o\rangle^3}{\langle\lambda\kappa\rangle} \mathrm{e}^{\frac{i\omega_i[\mu\tilde{\kappa}]}{\langle\lambda o\rangle}},$$
- On $\{\langle\lambda 0\rangle \neq 0\}$, $\bar{\partial} g_{\omega,\kappa,\tilde{\kappa}} = h_\kappa$ so $Q_{g_\kappa} = V_{h_\kappa}$ (cf. Sharma's light-ray transform).
- Mellin transform to $\Delta = k \in \mathbb{Z}_{\leq 2} \leftrightarrow$ soft expansion in $\omega \rightsquigarrow$
$$g^k = \frac{\mathrm{i}^{2-k}}{(2-k)!} \frac{[\mu\,\tilde{\kappa}]^{2-k}\langle\lambda o\rangle^{k+1}}{\langle\lambda\,\kappa\rangle}.$$

Expands to give Strominger's $Lw_{1+\infty}$-algebra of symmetries complete with combinatoric factors:

- $Lw_{1+\infty} \leftrightarrow$ Hamiltonians on the $\mu^{\dot\alpha}$-plane, Laurent in λ:
$$w^p_{m,r} = (\mu^{\dot 0})^{p-m-1}(\mu^{\dot 1})^{p+m-1}\frac{\langle\lambda\iota\rangle^r}{\langle\lambda o\rangle^{r+2p-4}}, \qquad |m| \leq p-1$$
- Poisson brackets
$$\{w^p_{m,r}, w^q_{n,s}\} = (2(p-1)n - 2(q-1)m)w^{p+q-2}_{m+n,r+s}.$$

Alternate formulations

- Beyond SD sector, ideas embed into 4d ambitwistor-string, still non-trivially have
$$V_g \cdot V_{g'} \sim \frac{1}{\sigma - \sigma'} V_{\{g,g'\}} + \dots .$$
but now $V_g \cdot V_{\tilde{g}}$ more complicated, but computable.
- Story extends to $\Lambda \neq 0$, YM and nonlinear backgrounds.
- Atul Sharma has full off-shell twistor action for GR [2104.07031].
- Split signature formulation requires 'open string' approach, cf. old papers |LeBrun & M. 2002,2004, 2005 2010].

Quantization

- Einstein gravity tree $=$ tree sigma model correlator (MHV).
- Does full quantum sigma model correlator $\leftrightarrow$ gravity loops?

$$\langle 1\,2\rangle^{2n} \prod_{i=3}^{n} \frac{1}{\langle 1\,i\rangle^2 \langle 2\,i\rangle^2} \exp\left[-\frac{\mathrm{i}\,\alpha}{8\pi} \sum_{j\neq i} \frac{[i\,j]}{\langle i\,j\rangle} \frac{\langle 1\,i\rangle^2 \langle 2\,j\rangle^2}{\langle 1\,2\rangle^2} \right] .$$

- Does quantum sigma model realize $W_{1+\infty}$ or W-gravity?
- Moyal quantization of $\mu^{\dot{\alpha}}$-plane and ‘palatial twistors’?

Conclusions:

- Penrose's nonlinear graviton realizes:
  $\{$space of SD gravitons $\} = \{$ loop group for $Lw_{1+\infty}\}$.
- Gravity tree amplitudes generated by on-shell action of twistor sigma model ($\leftrightarrow$ cuts of $\mathscr{I}$).
  Explains tree formulae of [Bern, Dixon, Perelstein, Rosowsky 1998].
- Gives value of Einstein-Hilbert action at MHV.
- $Lw_{1+\infty}$ acts on $\mathbb{PT}$ by Čech vertex ops for SD gravitons.
- Soft graviton expansion $\leftrightarrow$ Taylor expansion for $Lw_{1+\infty}$.

Questions:

- $N = 8$ formulation with S_n symmetry?
- Axiomatize correspondence between celestial OPES and twistor sigma model/ambitwistor string OPEs
  (cf. Eduardo's talk).

Thank You!