Carroll: to run or to stand still

Jelle Hartong

University of Edinburgh, School of Mathematics

Celestial Sphere: holography, CFT and amplitudes

Higgs Centre

14 September 2021

In collaboration with:

Jan de Boer (University of Amsterdam), Niels Obers (Nordita & Niels Bohr Institute)

Watse Sybesma (University of Iceland), Stefan Vandoren (Utrecht University)

THE ROYAL SOCIETY

LEVERHULME TRUST

Introduction

- The Carroll limit is the speed of light to zero contraction of the Poincaré group. [Lévy-Leblond, 1965]
- In this limit you can 'run' (boost yourself) without moving in space.
- Reminiscent of the Red Queen's race from Lewis Carroll's Through the Looking-Glass.
- What happens when we expand a relativistic theory around $c = 0$ and is it good for anything?

Introduction

- The Carroll group is a kinematical group and it is possible to define Carrollian manifolds.
- Carrollian manifolds admit vielbeine that transform under local Carroll boosts (as opposed to local Lorentz boosts). [Bekaert, Morand, 2015], [JH, 2015], [Figueroa-O'Farrill, Prohazka, 2018]
- Null hypersurfaces are examples of Carrollian manifolds and this includes null infinity of asymptotically flat spacetime. [Duval, Gibbons, Horvathy, 2014]

Introduction

- An incomplete list of examples where Carroll symmetries emerge:
  - black hole membrane paradigm [Donnay, Marteau, 2019], [Penna, 2018]
  - ‘flat space holography’ (Carroll perspective so far only in 3D) [Bagchi, Detournay, Fareghbal, Simón, 2012], [JH, 2015], [Ciambelli, Marteau, Petkou, Petropoulos, Siampos, 2018]
  - tensionless limits of strings [Bagchi, 2013]
  - limits of GR [Henneaux, 1979], [Bergshoeff, Gomis, Rollier, ter Veldhuis, 2017]
  - Inflationary cosmology [de Boer, JH, Obers, Sybesma, Vandoren, to appear]
  - and generally whenever there is an effective speed of light that is much smaller than the velocity of concern

Outline

- Carroll symmetries
- Field theories
- Fluids and cosmology
- Geometry: $c = 0$ expansions and null hypersurfaces
- Null infinity and a boundary stress tensor for $\mathcal{I}^+$ in 3D

The Carroll limit

- Lorentz transformations with parameter $\vec{\beta}$:

$$\begin{aligned} ct' &= \gamma(ct - \vec{\beta}\cdot\vec{x})\,, \qquad \gamma = (1-\vec{\beta}^2)^{-1/2} \\ \vec{x}'_{\|} &= \gamma(\vec{x}_{\|} - \vec{\beta}ct)\,, \qquad \vec{x}'_{\perp} = \vec{x}_{\perp} \end{aligned}$$

- Carroll limit: $\vec{\beta} = c\vec{b}$, rescale $c \to \varepsilon c$ and $\varepsilon \to 0$ with $\vec{b}$ fixed.

Carroll transformation: $$t' = t - \vec{b}\cdot\vec{x}\,, \qquad \vec{x}' = \vec{x}$$

- Space is absolute and time is relative.
- No Lorentz contraction or time dilation as $\gamma \to 1$ in the Carroll limit.

The Carroll limit

- If a Carroll observer measures time and space differences Δt and $\Delta \vec{x}$ between two events, then a boosted Carroll observer measures the same distance, but a time difference $\Delta t' = \Delta t - \vec{b} \cdot \Delta \vec{x}$.
- If $\vec{b}$ is large enough $\Delta t' < 0$ while $\Delta t > 0$, i.e. two observers do not necessarily agree on which event happened first.
- Coordinate time is not a good clock to describe the motion of a particle. Instead we use proper time, the affine parameter along the worldline.
- Velocities transform by rescaling $\vec{v}' = \frac{d\vec{x}'}{dt'} = \frac{\vec{v}}{1 - \vec{b} \cdot \vec{v}}$
- $\vec{v} = 0$ and $\vec{v} \neq 0$ are not related by a Carroll boost: either you stand still or you always move.

Carroll metric

- Spatial distances are Carroll invariant:

$$ds^2 = -c^2 dt^2 + d\vec{x}^2 \to h = d\vec{x}^2$$

- At a fixed point in space you can measure time intervals.
- Limit of inverse Poincaré metric tells us that $v = \frac{\partial}{\partial t}$ is Carroll invariant.
- The light cone $-c^2 t^2 + \vec{x}^2 = 0$ becomes the line $\vec{x} = 0$ for all t: light is not moving in space!

Carroll algebra

- Lorentz transformation of energy and momentum:

$$E' = \gamma(E - c\vec{\beta}\cdot\vec{p})\,, \qquad \vec{p}'_{\|} = \gamma\left(\vec{p}_{\|} - \vec{\beta}\frac{E}{c}\right)\,, \qquad \vec{p}'_{\perp} = \vec{p}_{\perp}$$

- Carroll limit: $\vec{\beta} = c\vec{b}$, rescale $c \to \varepsilon c$ and $\varepsilon \to 0$ with $\vec{b}$ fixed.

Carroll transformation: $$E' = E\,, \qquad \vec{p}\,' = \vec{p} - \vec{b}E$$

- The Carroll algebra is spanned by H, P_i, C_i, J_{ij} with the nonzero brackets ($i, j = 1, \ldots, d$):

$$[P_i, C_j] = \delta_{ij} H\,, \qquad [J_{ij}, P_k] = 2\delta_{k[i}P_{j]}\,, \qquad [J_{ij}, C_k] = 2\delta_{k[i}C_{j]}$$
$$[J_{ij}, J_{kl}] = -2\delta_{i[k}J_{l]j} + 2\delta_{j[k}J_{l]i}$$

- The Hamiltonian is a central element.

Current conservation

- On shell conserved currents for a field theory with Carroll symmetries:

$$\partial_\mu \left(T^\mu{}_\nu K^\nu\right) = 0$$

- $T^\mu{}_\nu$ is the energy-momentum tensor and K^ν is one of the generators:

$$H = \partial_t\,, \qquad P_i = \partial_i\,, \qquad C_i = x^i \partial_t\,, \qquad J_{ij} = x^i \partial_j - x^j \partial_i$$

  These are the 'Killing' vectors of the Carroll metric data: $v = \partial_t$ and $h = \delta_{ij} dx^i dx^j$.

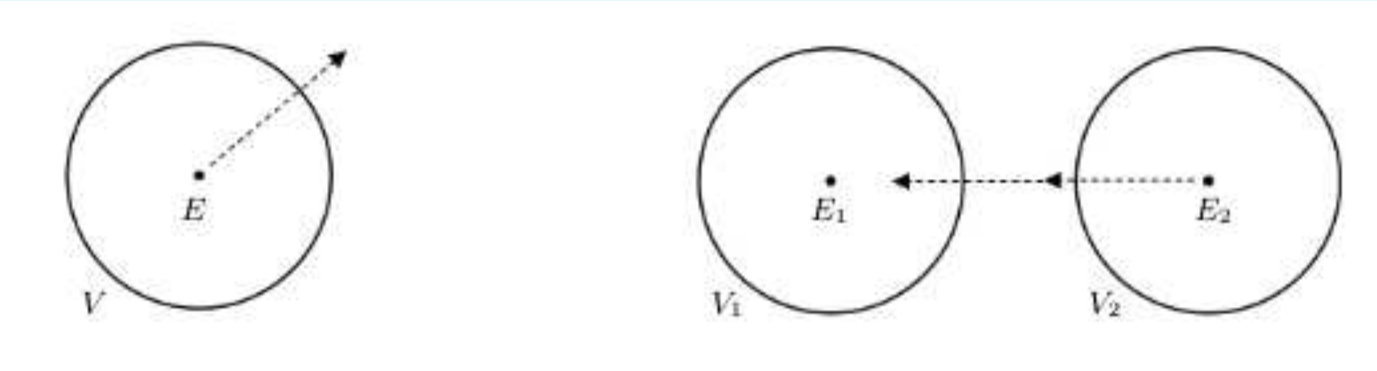
- This implies:

$$\partial_\mu T^\mu{}_\nu = 0\,, \qquad T^i{}_t = 0\,, \qquad T^i{}_j = T^j{}_i$$

- We conclude that the energy flux $T^i{}_t$ must vanish!

No energy flux

- The vanishing of the energy flux also follows from the $c \to 0$ limit of the relativistic property $\frac{1}{c}T^i{}_t + cT^t{}_i = 0$.
- It follows that $\partial_t T^0{}_0 = 0$ or $\frac{d}{dt}\int_V d^dx T^0{}_0 = 0$ for any volume V.
- Contrast this with $\frac{d}{dt}\int_V d^dx T^0{}_0 = -\int_{\partial V} d^{d-1}x n_i T^i{}_0$.
- Single particle: if the energy is nonzero it cannot move and if it can move the energy must be zero.



Irreps of the Carroll algebra ($d = 3$)

- Eigenstates of H and the quartic Casimir $W_i W_i$

$$W_i = H S_i + \varepsilon_{ijk} C_j P_k$$

- Consider energy-momentum eigenstates (E, p_i) of H and P_i.
- When $E \neq 0$ we can always go to a frame where $p_i = 0$ by performing a Carroll boost. In this case the little group is $SO(3)$ and the eigenvalues of $W_i W_i$ are $E^2 s(s+1)$ with $s = 0, 1/2, 1, \ldots$.
- When $E = 0$ the momentum p_i is Carroll boost invariant. Using a rotation we can WLOG set $\vec{p} = p\hat{e}_3$. On such states $W_i = \varepsilon_{ijk} C_j P_k$ so that $W_3 = 0$. The little group is $ISO(2)$ generated by W_1, W_2, L where $L = P_i S_i$ (helicity).

Carroll field theory

- Consider a relativistic field theory:

$$\mathcal{L} = \frac{1}{2c^2}\dot{\phi}^2 - \frac{1}{2}(\partial_i \phi)^2 - V(\phi)$$

- Sending $c \to 0$ (and rescaling $\mathcal{L}$) gives

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \tilde{V}(\phi)$$

  where $\tilde{V}$ is whatever is left of the potential in the limit.

- For $\tilde{V}$ a quadratic potential this corresponds to the $E \neq 0$ irrep.
- The energy flux vanishes due to missing gradient term.

Carroll field theory

- Rewrite the relativistic theory as

$$\mathcal{L} = \chi\dot{\phi} - \frac{c^2}{2}\chi^2 - \frac{1}{2}(\partial_i\phi)^2 - V(\phi)$$

- Sending $c \to 0$ leads to

$$\mathcal{L} = \chi\dot{\phi} - \frac{1}{2}(\partial_i\phi)^2 - \tilde{V}(\phi)$$

- The latter is Carroll boost invariant under

$$\delta\phi = \vec{b}\cdot\vec{x}\dot{\phi}\,, \qquad \delta\chi = \vec{b}\cdot\vec{x}\dot{\chi} + \vec{b}\cdot\vec{\partial}\phi$$

- χ is a Lagrange multiplier for $\dot{\phi} = 0$. This corresponds to the $E = 0$ irrep.

- The energy flux vanishes on shell due to the constraint $\dot{\phi} = 0$.

Electric Carroll

- Electric $c \to 0$ limit of Maxwell:

$$\mathcal{L} = \frac{1}{2} E_i E_i \,, \qquad E_i = \partial_i A_t - \partial_t A_i$$

- This is Carroll invariant under: $\delta A_t = \vec{b} \cdot \vec{x} \partial_t A_t$ and $\delta A_i = \vec{b} \cdot \vec{x} \partial_t A_i + b_i A_t$.
- Energy-momentum tensor:

$$T^t{}_t = -\frac{1}{2} E_i E_i \,, \quad T^i{}_t = 0 \,, \quad T^t{}_j = (\vec{E} \times \vec{B})_j \,, \quad T^i{}_j = -E_i E_j + \frac{1}{2} \delta_{ij} E^2$$

- EOM:

$$\partial_i B_i = 0 \,, \qquad \partial_t B_i + \left(\vec{\nabla} \times \vec{E} \right)_i = 0$$

$$\partial_i E_i = 0 \,, \qquad \partial_t E_i = 0$$ Ampère's law without $\vec{\nabla} \times \vec{B}$ term

Magnetic Carroll

- Magnetic $c \to 0$ limit of Maxwell:

$$\mathcal{L} = \chi_i E_i - \frac{1}{2} B_i B_i \,, \qquad E_i = \partial_i A_t - \partial_t A_i \,, \qquad B_i = \left(\vec{\nabla} \times \vec{A}\right)_i$$

- χ_i is a Lagrange multiplier transforming under Carroll boosts as $\delta\chi_i = \vec{b} \cdot \vec{x} \partial_t \chi_i + \left(\vec{b} \times \vec{B}\right)_i$.
- Energy-momentum tensor:

$$T^t{}_t = -\frac{1}{2} B_i B_i \,, \quad T^i{}_t = 0 \,, \quad T^t{}_j = (\vec{\chi} \times \vec{B})_j \,, \quad T^i{}_j = -B_i B_j + \frac{1}{2} \delta_{ij} B^2$$

- EOM (χ_i plays the role of the electric field):

$$\partial_i B_i = 0 \,, \qquad \partial_t B_i = 0 \; \text{ Faraday without } \vec{\nabla} \times \vec{E} \text{ term}$$

$$\partial_i \chi_i = 0 \,, \qquad \partial_t \chi_i - \left(\vec{\nabla} \times \vec{B}\right)_i = 0$$

Carroll fields in 2D

- In $1+1$ dimensions the Carroll algebra admits a central extension allowing for more interesting theories.
- For $i,j=1,\ldots,2n$ and ω_{ij} a constant antisymmetric invertible matrix consider

$$\mathcal{L}=\frac{1}{2}\partial_\tau X^i\partial_\tau X^j-\omega_{ij}X^i\partial_\sigma X^j$$

- This is Carroll invariant with $\delta X^i=b\sigma\partial_\tau X^i-b\tau\omega_{ij}X^j$.
- This model can be obtained as a gauged fixed version of a Polyakov-type theory for a closed string whose worldsheet is Carrollian. [Bidussi, Harmark, JH, Obers, Oling, to appear]

Carroll perfect fluids

- The most general perfect fluid is (in LAB frame) [de Boer, JH, Obers, Sybesma, Vandoren, 2017]

$$T^t{}_t = -\mathcal{E}\,, \qquad T^i{}_t = -\left(\mathcal{E}+P\right)v^i\,, \qquad T^t{}_j = \mathcal{P}_j\,, \qquad T^i{}_j = P\delta^i_j + v^i\mathcal{P}_j$$

- Momentum density $\mathcal{P}_i = \rho v^i$
- All functions depend on the fluid variables: T and v^i.
- From the transformation of $T^\mu{}_\nu$ under diffeos we conclude that $\mathcal{P}_i = \rho v^i$ transforms under a Carroll boost as

$$\mathcal{P}'_i = \rho' v'^i = \rho' \frac{v^i}{1-\vec{b}\cdot\vec{v}} = \rho v^i(1-\vec{b}\cdot\vec{v}) - b_i\left(\mathcal{E}+P\right)$$

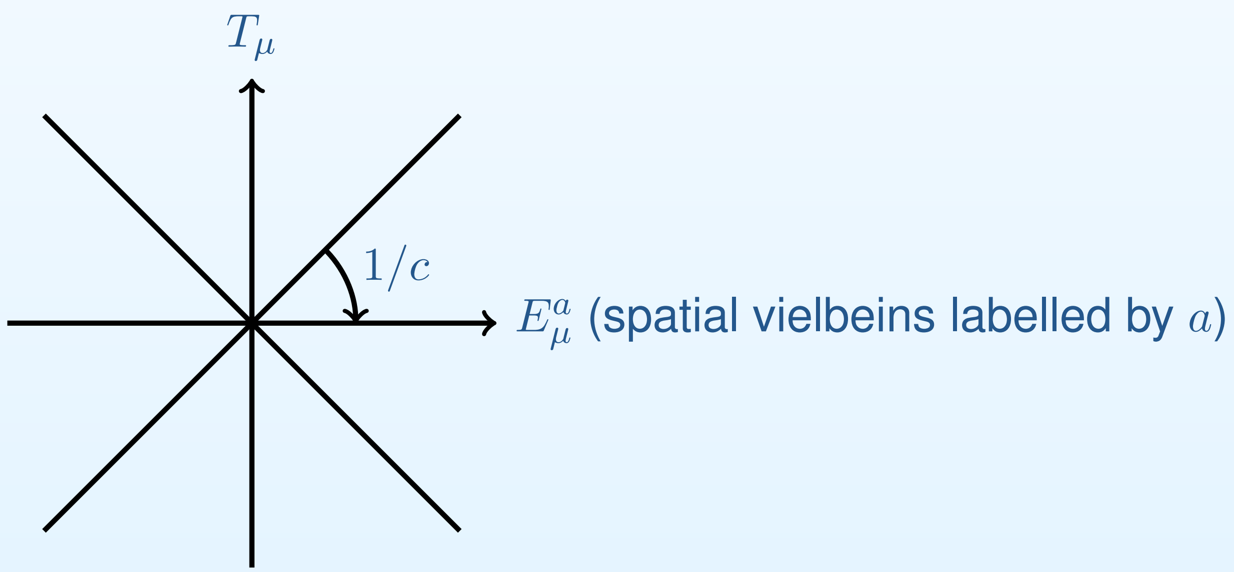
- Hence we need $\mathcal{E}+P=0$ for any Carroll fluid!
- Reminiscent of the equation of state in cosmology ($w=-1$).

Cosmology

- Hubble law: $v = Hd$
- Hubble radius: $R_H = cH^{-1}$
- If distances d are much larger than R_H we have $v \gg c$.
- super-Hubble scales are Carrollian
- As $c \to 0$, the Hubble radius vanishes, so the entire universe becomes super-Hubble, i.e. Carrollian.
- This is an ultra-local limit.
- As we expand away from $c = 0$, Hubble cells grow containing more and more d.o.f.
- Expanding inflationary solutions around $c = 0$ naturally leads to small slow roll parameters.

Expanding GR around $c = 0$

- A convenient way to make the c-dependence of GR manifest is to write $g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}$ and $g^{\mu\nu} = -\frac{1}{c^2} T^\mu T^\nu + \Pi^{\mu\nu}$.
- Signature of $\Pi_{\mu\nu}$ is $(0, 1, \ldots, 1)$.
- Light cones in tangent space have slope $1/c$:



Expanding GR around $c = 0$

- Expanding the vielbeine in a Taylor series in c^2 gives:

$$\begin{aligned} T_\mu &= \tau + O(c^2)\,, \qquad \Pi_{\mu\nu} = h_{\mu\nu} + c^2\Phi_{\mu\nu} + O(c^4) \\ T^\mu &= v^\mu + c^2 M^\mu + O(c^4)\,, \qquad \Pi^{\mu\nu} = h^{\mu\nu} + O(c^2) \end{aligned}$$

- Hence the metric and its inverse are:

$$\begin{aligned} g_{\mu\nu} &= h_{\mu\nu} + c^2\left(\Phi_{\mu\nu} - \tau_\mu\tau_\nu\right) + O(c^4) \\ g^{\mu\nu} &= -\frac{1}{c^2}v^\mu v^\nu + h^{\mu\nu} - 2v^{(\mu}M^{\nu)} + O(c^2) \end{aligned}$$

- Expanding the local Lorentz transformation that act on T_μ and $\Pi_{\mu\nu}$ in c^2 we obtain the local Carroll transformations ($\tau_\mu\lambda^\mu = 0$):

$$\delta\tau_\mu = h_{\mu\nu}\lambda^\nu\,, \qquad \delta M^\mu = \lambda^\mu\,, \qquad \delta h^{\mu\nu} = 2v^{(\mu}\lambda^{\nu)}\,, \qquad \delta\Phi_{\mu\nu} = 2\tau_{(\mu}\lambda_{\nu)}$$

- The measure expands as $\sqrt{-g} = ce\left(1 + O(c^2)\right)$ where $e = \det\left(\tau_\mu\,, e_\mu^a\right)$ with $h_{\mu\nu} = \delta_{ab} e_\mu^a e_\nu^b$ $(a, b = 1, \ldots, d)$.
- Einstein–Hilbert plus matter: $\mathcal{L} = \frac{c^3}{16\pi G}\sqrt{-g}R + \mathcal{L}_{\text{mat}}$
- It can be shown that $R = O(c^{-2})$ so that $\mathcal{L}_{\text{mat}}$ must be $O(c^2)$.
- In order that $\delta\mathcal{L}_{\text{mat}} = \frac{1}{2c}\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} = O(c^2)$ we must have

$$T^{\mu\nu} = -\mathcal{T} v^\mu v^\nu + c^2 \mathcal{T}^{\mu\nu} + O(c^4) \qquad \text{and then}$$

$$\delta\mathcal{L}_{\text{mat}} = c^2 e\left(-\mathcal{T} v^\mu \delta\tau_\mu + \frac{1}{2}\left(\mathcal{T}^{\mu\nu} + 2\mathcal{T} v^{(\mu} M^{\nu)}\right)\delta h_{\mu\nu} + O(c^2)\right)$$

- The Carroll EMT is then

$$\mathcal{T}^\mu{}_\nu = -\mathcal{T} v^\mu \tau_\nu + \left(\mathcal{T}^{\mu\rho} + 2\mathcal{T} v^{(\mu} M^{\rho)}\right) h_{\rho\nu}$$

- Agrees with known examples and obeys the no energy flux condition: $h_{\mu\rho} v^\nu \mathcal{T}^\mu{}_\nu = 0$.

Null hypersurfaces

- The metric structure on any null hypersurface is Carrollian.
- Most general metric for which $u = \text{cst}$ is a hypersurface:

$$ds^2 = 2du\,(\Phi du - \hat{\tau}_\mu dx^\mu) + h_{\mu\nu} dx^\mu dx^\nu$$

$$\Phi = -\tau_\mu M^\mu + \frac{1}{2} h_{\mu\nu} M^\mu M^\nu \,, \qquad \hat{\tau}_\mu = \tau_\mu - h_{\mu\nu} M^\nu$$

- Local Lorentz transformations that leave the normal 1-form du invariant (null rotations) lead to local Carroll transformations.
- A sufficient condition for the ambient EMT $T^M{}_N$ to induce Carroll transport on the null hypersurface is $T^u{}_\nu = 0$:

$$0 = \nabla_M T^M{}_\nu = \overset{C}{\nabla}_\mu T^\mu{}_\nu - 2\overset{C}{\Gamma}{}^\mu_{[\mu\rho]} T^\rho{}_\nu + 2\overset{C}{\Gamma}{}^\sigma_{[\rho\nu]} T^\rho{}_\sigma$$

for some suitably chosen Carrollian connection $\overset{C}{\nabla}_\mu$.

3D Asymptotically flat spaces

- Minkowski space-time in EF coordinates: $ds^2 = -du^2 - 2dudr + r^2 d\varphi^2$; u is retarded time, r parameter of null geodesics, φ angular coordinate.
- Asymptotically flat space-time in BMS gauge (large r expansion) [Barnich, Compère, 2006]:

$$
\begin{aligned}
g_{rr} &= r^{-2} h_{rr} + \mathcal{O}(r^{-3})\,, & g_{uu} &= h_{uu} + \mathcal{O}(r^{-1})\,, \\
g_{ru} &= -1 + r^{-1} h_{ru} + \mathcal{O}(r^{-2})\,, & g_{u\varphi} &= h_{u\varphi} + \mathcal{O}(r^{-1})\,, \\
g_{r\varphi} &= h_1(\varphi) + r^{-1} h_{r\varphi} + \mathcal{O}(r^{-2})\,, & g_{\varphi\varphi} &= r^2 + r h_{\varphi\varphi} + \mathcal{O}(1)\,.
\end{aligned}
$$

- Most general Taylor expansion for a flat boundary at null infinity in 3D.

- We generalize this by allowing for arbitrary sources: Φ, $\hat{\tau}_\mu$, $h_{\mu\nu}$ (vanishing determinant).

$$
\begin{aligned}
g_{rr} &= 2\Phi r^{-2} + \mathcal{O}(r^{-3}) \,, \\
g_{r\mu} &= -\hat{\tau}_\mu + r^{-1} h_{(1)r\mu} + \mathcal{O}(r^{-2}) \,, \\
g_{\mu\nu} &= r^2 h_{\mu\nu} + r h_{(1)\mu\nu} + h_{(2)\mu\nu} + \mathcal{O}(r^{-1}) \,.
\end{aligned}
$$

- In terms of vielbeine $ds^2 = -2UV + EE$ the metric boundary conditions are:

$$
\begin{aligned}
U_r &= 1 + \mathcal{O}(r^{-1}) \,, & V_\mu &= \tau_\mu + \mathcal{O}(r^{-1}) \\
U_\mu &= r U_{(1)\mu} + \mathcal{O}(1) \,, & E_r &= r^{-1} e_\nu M^\nu + \mathcal{O}(r^{-2}) \\
V_r &= r^{-2} \tau_\mu M^\mu + \mathcal{O}(r^{-3}) \,, & E_\mu &= r e_\mu + \mathcal{O}(1)
\end{aligned}
$$

- Relation to the metric sources:

$$
h_{\mu\nu} = e_\mu e_\nu \,, \qquad \hat{\tau}_\mu = \tau_\mu - e_\mu e_\nu M^\nu \,, \qquad \Phi = -\tau_\mu M^\mu + \frac{1}{2} (e_\mu M^\mu)^2
$$

Null Infinity is described by Carrollian geometry

- Consider bulk local Lorentz transformations that keep the normal U fixed. These act on the boundary vielbeins as Carroll boosts, i.e.

$$e'_\mu = e_\mu \,, \qquad \tau'_\mu = \tau_\mu + \lambda e_\mu \,, \qquad M'^\mu = M^\mu + \lambda e^\mu + \frac{1}{2}\lambda^2 v^\mu \,.$$

- Together with near boundary bulk diffeomorphisms these generate all the local symmetries acting on the sources τ_μ, e_μ and M^μ.

- It can be shown that the M^μ source is pure gauge.

Well-posed variational problem

- Bulk plus Gibbons–Hawking boundary terms at $\mathcal{I}^+$:

$$S = \int d^3x\sqrt{-g}R + \alpha\int_{\mathcal{I}^+}\frac{1}{2}\epsilon_{MNP}dx^M \wedge dx^N V^P \left(E^R E^S \nabla_R U_S\right)$$

- The GH term at $\mathcal{I}^+$ is the unique term that is invariant under: i). bulk local Lorentz transformations that leave U invariant and ii). bulk local Lorentz transformations that act as $\delta U_M = \bar{\lambda}U_M$, and $\delta V_M = -\bar{\lambda}V_M$. The last symmetry is special for null hypersurface orthogonal vectors.

- We will demand that δS is finite i.e. $\mathcal{O}(1)$ in r and that it is zero when the variations of the sources vanish at $\mathcal{I}^+$.

- Variation of the bulk action:

$$\delta S_{\text{bulk}} = -\frac{1}{2}\int_{\partial\mathcal{M}} \epsilon_{MNP} dx^M \wedge dx^N V^P U_Q J^Q$$

where $J^P = g^{MN}\delta\Gamma^P_{MN} - g^{MP}\delta\Gamma^N_{NM}$.

- No counterterm to cancel leading divergence at r^2. Need to set $\partial_\mu e_\nu - \partial_\nu e_\mu = 0$ to remove divergence.
- We then find at $O(1)$:

$$\frac{1}{2}\epsilon_{MNP} dx^M \wedge dx^N V^P U_Q J^Q|_{\partial\mathcal{M}} = e d^2x\Big(-\mathcal{T}^\mu \delta\tau_\mu + \frac{1}{2}\mathcal{T}^{\mu\nu}\delta h_{\mu\nu} + \mathcal{O}(r^{-1})\Big)$$

- We thus do not need the GH boundary term, i.e. $\alpha = 0$.

Well-posed variational problem

- Local Carroll boost invariance leads to $h_{\mu\rho}v^\nu \mathcal{T}^\mu{}_\nu = 0$.
- Demanding invariance under boundary diffeos we find the Ward identity:

$$\overset{c}{\nabla}_\mu \mathcal{T}^\mu{}_\nu - 2\overset{c}{\Gamma}{}^\mu_{[\mu\rho]}\mathcal{T}^\rho{}_\nu + 2\overset{c}{\Gamma}{}^\rho_{[\mu\nu]}\mathcal{T}^\mu{}_\rho = 0$$

  where we defined $\mathcal{T}^\mu{}_\nu = -\mathcal{T}^\mu\tau_\nu + \mathcal{T}^{\mu\rho}h_{\rho\nu}$.
- Hit the diffeo Ward identity with any vector K:

$$e^{-1}\partial_\mu\left(eK^\nu\mathcal{T}^\mu{}_\nu\right) + \mathcal{T}^\mu\mathcal{L}_K\tau_\mu - \frac{1}{2}\mathcal{T}^{\mu\nu}\mathcal{L}_K h_{\mu\nu} = 0$$

BMS Symmetries

- When the boundary is flat any solution to

$$\mathcal{L}_K \tau_\mu = \Omega \tau_\mu + h_{\mu\nu} \zeta^\nu \,, \qquad \mathcal{L}_K h_{\mu\nu} = 2\Omega h_{\mu\nu}$$

gives rise to a conserved current.

- Here $v^\mu \partial_\mu \Omega = 0$ due to the constraint $\partial_\mu e_\nu - \partial_\nu e_\mu = 0$. Recall $h_{\mu\nu} = e_\mu e_\nu$.

- The resulting ‘Killing’ vectors K are

$$\begin{aligned} K^\varphi &= f(\varphi) \,, \qquad K^u = f'(\varphi) u + g(\varphi) \,, \\ \Omega &= f'(\varphi) \,, \qquad \zeta^u = 0 \,, \qquad \zeta^\varphi = f''(\varphi) u + g'(\varphi) \,. \end{aligned}$$

which generate the BMS algebra.

Outlook

- Carroll strings
- Tensionless strings
- 4D asymptotically flat spacetimes
- Expansions around $c = 0$ and cosmology
- Carroll fluids: applications to supersonic behaviour?