Twistors, light transforms, and the celestial Grassmannian

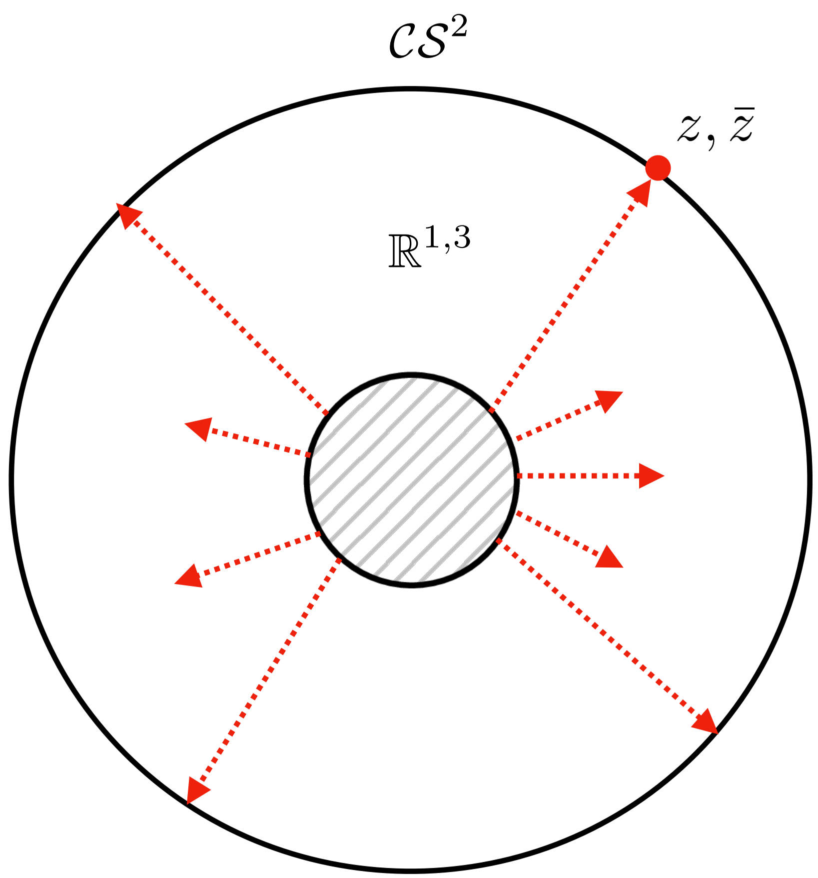
Edinburgh, 2021

Atul Sharma
Mathematical Institute
Oxford

Based on [2107.06250]

Locality in the sky?

[Pasterski, Shao '17]



Momentum Eigenstates

Boost Eigenstates

$$|\omega, z, \bar{z}, \ell, \epsilon\rangle \longrightarrow |z, \bar{z}, h, \bar{h}, \epsilon\rangle = \int_0^\infty \frac{\mathrm{d}\omega}{\omega}\, \omega^\Delta\, |\omega, z, \bar{z}, \ell, \epsilon\rangle$$

$$h = \frac{\Delta + \ell}{2}\,,\ \bar{h} = \frac{\Delta - \ell}{2}$$

$$\epsilon = \pm 1$$

$$\mathcal{O}^{\epsilon}_{h,\bar{h}}(z, \bar{z})$$

Local?

Shadow transform

$$\mathbf{S}[\mathcal{O}^{\epsilon}_{h,\bar{h}}](z,\bar{z}) = \int_{\mathbb{C}} \frac{\mathrm{d}^2 w\ \mathcal{O}^{\epsilon}_{h,\bar{h}}(w,\bar{w})}{(w-z)^{2-2h}\,(\bar{w}-\bar{z})^{2-2\bar{h}}}$$

Bulk dual state

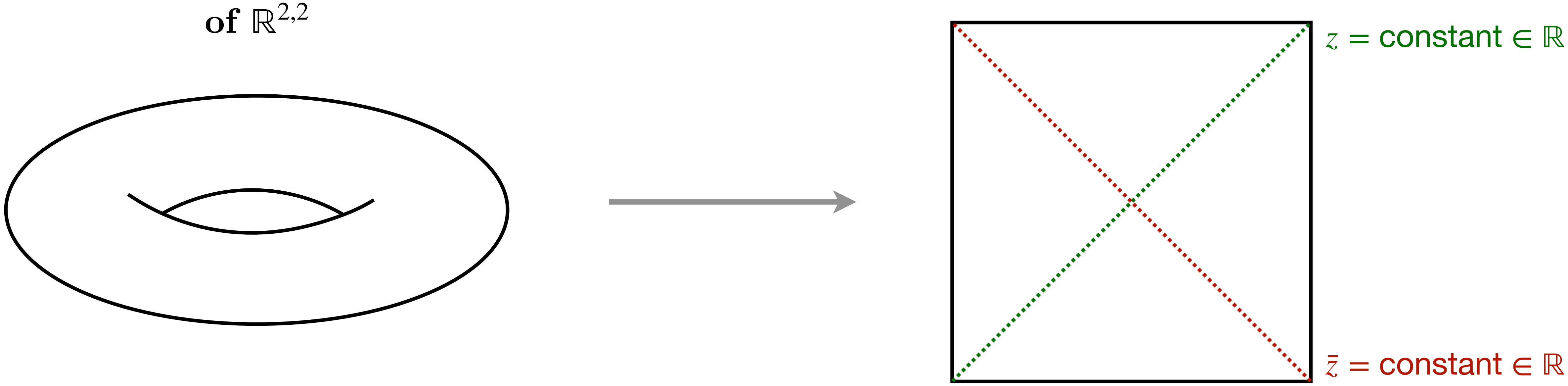
$$|z, \bar{z}, 1-h, 1-\bar{h}, \epsilon\rangle_{\mathbf{S}}$$

Light transforms

[Atanasov, Ball, Melton, Raclariu, Strominger '21]

Celestial torus of $\mathbb{R}^{2,2}$

$$ds^2 = dz\, d\bar{z}$$



		Bulk dual state
Lights transforms	$\mathbf{L}[\mathcal{O}^{\epsilon}_{h,\bar{h}}](z,\bar{z}) = \int_{\mathbb{R}} \frac{dw\ \mathcal{O}^{\epsilon}_{h,\bar{h}}(w,\bar{z})}{(w-z)^{2-2h}}$	$\lvert z,\bar{z},1-h,\bar{h},\epsilon\rangle_{\mathbf{L}}$
	$\bar{\mathbf{L}}[\mathcal{O}^{\epsilon}_{h,\bar{h}}](z,\bar{z}) = \int_{\mathbb{R}} \frac{d\bar{w}\ \mathcal{O}^{\epsilon}_{h,\bar{h}}(z,\bar{w})}{(\bar{w}-\bar{z})^{2-2\bar{h}}}$	$\lvert z,\bar{z},h,1-\bar{h},\epsilon\rangle_{\bar{\mathbf{L}}}$

[Atanasov, Melton, Raclariu, Strominger '21] [Kravchuk, Simmons-Duffin '18]

Previously in celestial news

- Shadow transform $\longrightarrow$ adjoint operation in 2d CCFT.

  [Crawley, Miller, Narayanan, Strominger '21]

- Shadow transforms used for gluon conformal block expansions.

  [Fan, Fotopoulos, Stieberger, Taylor, Zhu '21]

- Light transform exchanges observed in scalar conformal block expansions.

  [Atanasov, Melton, Raclariu, Strominger '21]

- Light transform unravels symmetries of self-dual CCFT.

  [Himwich, Pate, Singh '21] [Strominger '21] [Guevara, Himwich, Pate, Strominger '21]

Today...

- From twistors to light transforms

- Light transformed celestial amplitudes

- Toward a Grassmannian formulation

Twistor eigenstates

Spinor-helicity

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}\,, \qquad \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}} \in \mathbb{R}^2$$

Real and independent in $\mathbb{R}^{2,2}$

Witten's 1/2 Fourier transform

$$|Z,\ell\rangle_{\mathbb{T}} \equiv |\lambda,\mu,\ell\rangle_{\mathbb{T}} := \int_{\mathbb{R}^2} \mathrm{d}^2\bar{\lambda}\; \mathrm{e}^{\mathrm{i}[\mu\,\bar{\lambda}]}\, |\lambda,\bar{\lambda},\ell\rangle$$

Twistor eigenstate

Momentum eigenstate

[Witten '03]

$$\mu^{\dot{\alpha}} \xleftrightarrow{\text{Fourier conjugate}} \bar{\lambda}_{\dot{\alpha}}$$

Little group

$$|r\,Z,\ell\rangle_{\mathbb{T}} = r^{-2\ell-2}\,|Z,\ell\rangle_{\mathbb{T}}\,, \qquad r \in \mathbb{R}^*$$

Twistors

$$Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha) \in \mathbb{RP}^3$$

Ambidextrous twistor amplitudes

$$\boxed{\mathcal{M}_n = \int \prod_{\bar{\mathsf{a}}\in -} \mathrm{d}^2\bar{\lambda}_{\bar{\mathsf{a}}}\ \mathrm{e}^{\mathrm{i}[\mu_{\bar{\mathsf{a}}}\,\bar{\lambda}_{\bar{\mathsf{a}}}]} \prod_{\mathsf{a}\in +} \mathrm{d}^2\lambda_{\mathsf{a}}\ \mathrm{e}^{\mathrm{i}\langle\bar{\mu}_{\mathsf{a}}\,\lambda_{\mathsf{a}}\rangle}\ A_n(\lambda_i,\bar{\lambda}_i)\,\delta^4\Big(\sum_{i=1}^{n}\lambda_i\bar{\lambda}_i\Big)}$$

Transform to twistor space

Transform to dual twistor space

[Arkani-Hamed, Cachazo, Cheung, Kaplan '09]

[Mason, Skinner '09]

Utilities

- Parity-symmetric prescription with cleanest results.
- Provides a natural home for BCFW recursion and on-shell diagrams.
- Inverse transforms give rise to Grassmannian formulae.

Twistors for AdS_3

Incidence relations

$$\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha$$

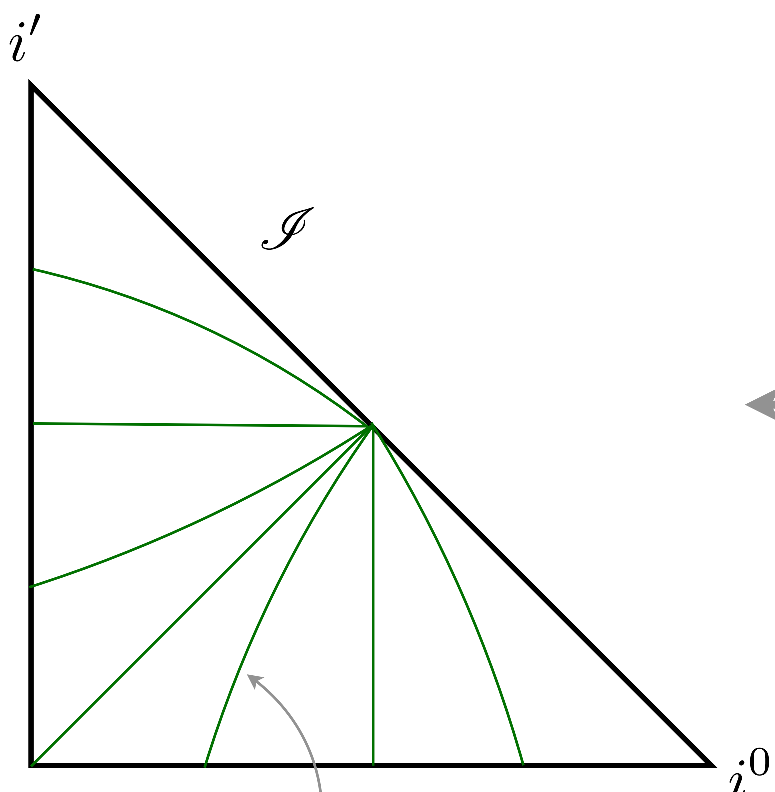
Space-time dilatations $\quad x^{\alpha\dot\alpha} \mapsto r\, x^{\alpha\dot\alpha} \quad \longleftrightarrow \quad$ **Twistor space dilatations** $\quad \Longrightarrow \quad \mu^{\dot\alpha} \mapsto r\,\mu^{\dot\alpha}$

Minitwistor space of AdS_3

$$= \mathbb{RP}^1 \times \mathbb{RP}^1 = \frac{\{\mu^{\dot\alpha} \in \mathbb{R}^2 - \{0\}\}}{\mu^{\dot\alpha} \sim r\,\mu^{\dot\alpha}} \times \frac{\{\lambda_\alpha \in \mathbb{R}^2 - \{0\}\}}{\lambda_\alpha \sim s\,\lambda_\alpha}$$

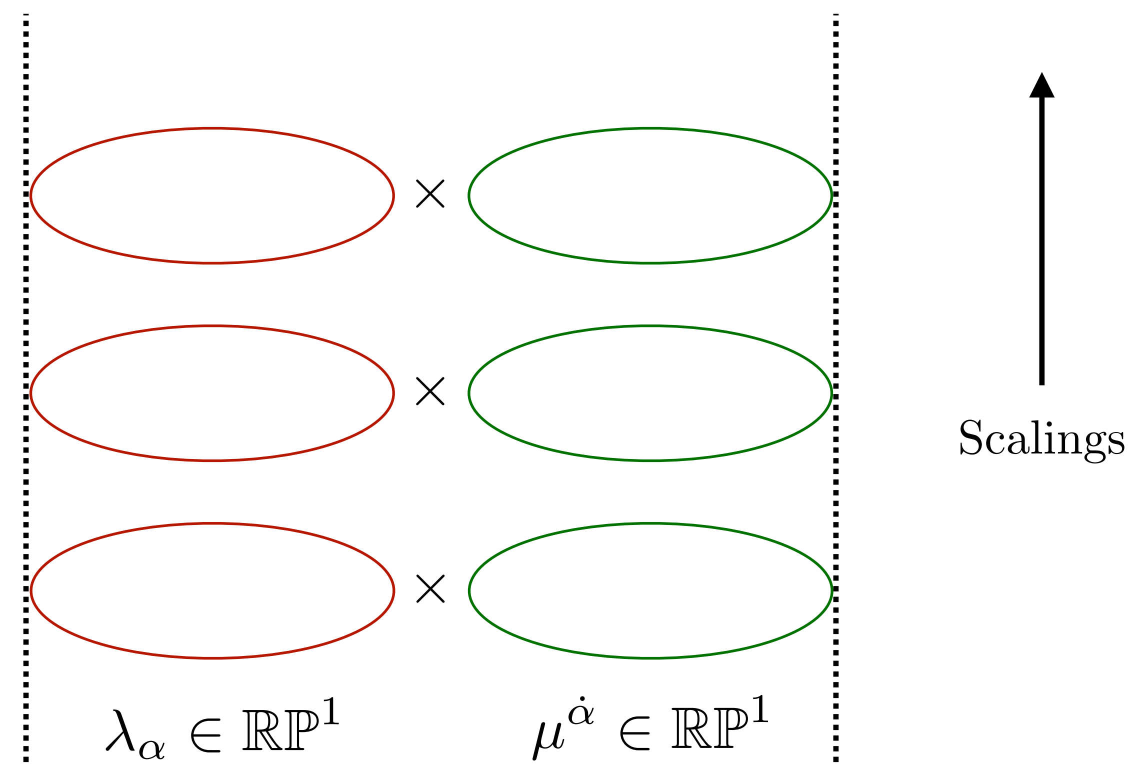
[Hitchin '82]

[Jones, Tod '85]



$\mathbb{R}^{2,2}$

AdS_3 foliation



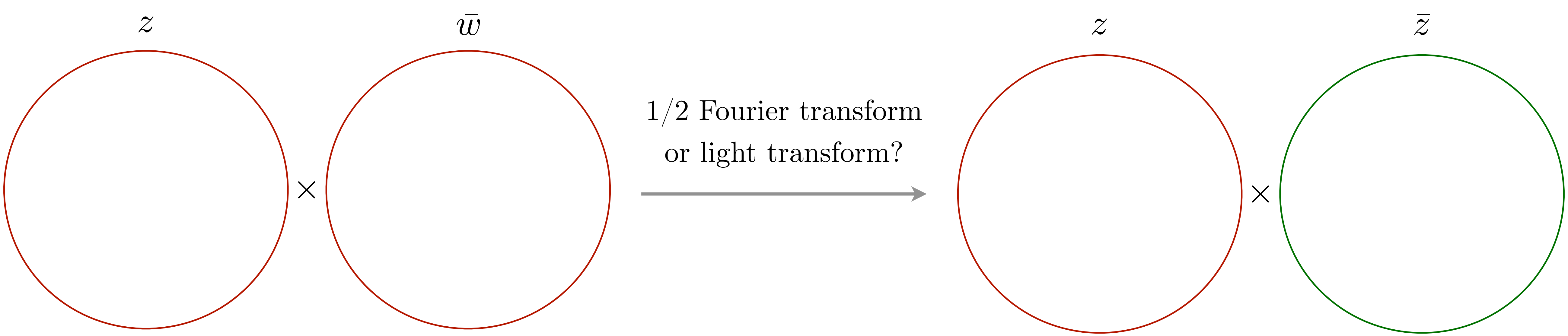
$$\mathbb{RP}^3 - (\{\lambda_\alpha = 0\} \cup \{\mu^{\dot\alpha} = 0\})$$

Primary twistor eigenstates

$$(\mu^{\dot\alpha}, \lambda_\alpha) \sim r\,(\mu^{\dot\alpha}, \lambda_\alpha) \xrightarrow{\text{Scale fixing}} \mu^{\dot\alpha} = \begin{pmatrix} -1 \\ \bar z \end{pmatrix}^{\dot\alpha}, \quad \lambda_\alpha = \epsilon\,\omega \begin{pmatrix} z \\ 1 \end{pmatrix}_\alpha$$

Conformal primary twistor eigenstate

$$|\underbrace{z, \bar z}_{\text{Point on twistor space of AdS}_3\text{ slices}}, \underbrace{h, 1-\bar h}_{\text{Transforms as conformal primary with these weights}}, \epsilon\rangle_{\mathbb{T}} := \underbrace{\int_0^\infty \frac{\mathrm{d}\omega}{\omega}}_{\text{“1/2 Mellin transform”}} \omega^{2h}\, |\lambda, \mu, \ell\rangle_{\mathbb{T}} \stackrel{?}{=} |z, \bar z, h, 1-\bar h, \epsilon\rangle_{\bar{\mathbf{L}}}$$



(Considered as an auxiliary copy of the celestial torus.)

Twistors to light transforms

$$
\boxed{
\begin{aligned}
&|z, \bar{z}, h, 1-\bar{h}, \epsilon\rangle_{\mathbb{T}} \\
&\qquad = \epsilon^{-2\ell}\,\Gamma(2-2\bar{h})\left(\mathrm{e}^{\pi \mathrm{i}(1-\bar{h})}\,|z, \bar{z}, h, 1-\bar{h}, \epsilon\rangle_{\bar{\mathbf{L}}} + \mathrm{e}^{\pi \mathrm{i}\bar{h}}\,|z, \bar{z}, h, 1-\bar{h}, -\epsilon\rangle_{\bar{\mathbf{L}}}\right)
\end{aligned}
}
$$

$$
\begin{tikzcd}
\textbf{Momentum eigenstates} \arrow[r, "1/2 \text{ Fourier transform}"] \arrow[d, "\text{Mellin transform}"'] & \textbf{Twistor eigenstates} \arrow[d, "1/2 \text{ Mellin transform}"] \\
\textbf{Boost eigenstates} \arrow[r, "\text{Light transforms}"] & \textbf{Primary twistor eigenstates}
\end{tikzcd}
$$

Motivates ambidextrous prescription for light transforms!

Ambidextrous light transforms

[Pasterski, Shao, Strominger '17]

$$\mathcal{A}_n(z_i,\bar{z}_i,\Delta_i,\ell_i,\epsilon_i) = \int_{\mathbb{R}^n_+} \prod_{j=1}^{n} \frac{\mathrm{d}\omega_j}{\omega_j}\,\omega_j^{\Delta_j}\; A_n(\epsilon_i\omega_i q_i,\ell_i)\,\delta^4\Bigg(\sum_{j=1}^{n}\epsilon_i\omega_i q_i\Bigg)$$

Celestial amplitude

Momentum space amplitude

$$q_{i\,\alpha\dot{\alpha}} = \begin{pmatrix} z_i\bar{z}_i & z_i \\ \bar{z}_i & 1 \end{pmatrix}_{\alpha\dot{\alpha}}$$

$$\epsilon_i = \pm 1$$

$$\mathcal{L}_n(z_i,\bar{z}_i,\Delta_i,\ell_i,\epsilon_i) = \int_{\mathbb{R}^n} \prod_{\bar{\mathsf{a}}} \frac{\mathrm{d}\bar{w}_{\bar{\mathsf{a}}}}{(\bar{w}_{\bar{\mathsf{a}}}-\bar{z}_{\bar{\mathsf{a}}})^{2-2\bar{h}_{\bar{\mathsf{a}}}}} \prod_{\mathsf{a}} \frac{\mathrm{d}w_{\mathsf{a}}}{(w_{\mathsf{a}}-z_{\mathsf{a}})^{2-2h_{\mathsf{a}}}} \times \mathcal{A}_n(z_{\bar{\mathsf{a}}},\bar{w}_{\bar{\mathsf{a}}},\Delta_{\bar{\mathsf{a}}},-,\epsilon_{\bar{\mathsf{a}}}\,;\,w_{\mathsf{a}},\bar{z}_{\mathsf{a}},\Delta_{\mathsf{a}},+,\epsilon_{\mathsf{a}})$$

$\mathsf{a} \longrightarrow$ Positive helicity gluon/graviton

$\bar{\mathsf{a}} \longrightarrow$ Negative helicity gluon/graviton

Gluon examples

$$\mathcal{A}_n(i^{\epsilon_i}_{\Delta_i,\ell_i}) \equiv \mathcal{A}_n(z_i,\bar{z}_i,\Delta_i,\ell_i,\epsilon_i) \propto \delta(\beta), \qquad \beta = \mathrm{i}\sum_{j=1}^{n}(\Delta_j - 1)$$

2 points

$$\mathcal{A}_2(1^+_{\Delta_1,+1}, 2^-_{\Delta_2,-1}) = \delta(\beta)\,\delta(z_{12})\,\delta(\bar{z}_{12})$$

$$z_{ij} = z_i - z_j$$

$$\implies \mathcal{L}_2(1^+_{\Delta_1,+1}, 2^-_{\Delta_2,-1}) = \delta(\beta)\int_{\mathbb{R}^2}\frac{\mathrm{d}w_1\,\mathrm{d}\bar{w}_2\;\delta(w_1 - z_2)\,\delta(\bar{z}_1 - \bar{w}_2)}{(w_1 - z_1)^{1-\Delta_1}\,(\bar{w}_2 - \bar{z}_2)^{1-\Delta_2}}$$
$$= \frac{\delta(\beta)}{z_{21}^{1-\Delta_1}\;\bar{z}_{12}^{1-\Delta_2}}\,.$$

Gluon examples

3 points

$$\mathcal{A}_3(1^-_{\Delta_1,-1},2^-_{\Delta_2,-1},3^+_{\Delta_3,+1}) = \delta(\beta)\,\Theta\!\left(\frac{z_{13}}{z_{12}}\right)\Theta\!\left(\frac{z_{32}}{z_{12}}\right)\frac{\delta(\bar z_{13})\,\delta(\bar z_{23})}{z_{12}^{-\Delta_3}\,z_{32}^{2-\Delta_1}\,z_{13}^{2-\Delta_2}}$$

$$\begin{aligned}\mathcal{L}_3(1^-_{\Delta_1,-1},2^-_{\Delta_2,-1},3^+_{\Delta_3,+1}) &= \delta(\beta)\int_{\mathbb{R}^3}\frac{\mathrm{d}\bar w_1}{(\bar w_1-\bar z_1)^{1-\Delta_1}}\frac{\mathrm{d}\bar w_2}{(\bar w_2-\bar z_2)^{1-\Delta_2}}\frac{\mathrm{d}w_3}{(w_3-z_3)^{1-\Delta_3}}\\ &\quad\times\Theta\!\left(\frac{z_1-w_3}{z_1-z_2}\right)\Theta\!\left(\frac{w_3-z_2}{z_1-z_2}\right)\frac{\delta(\bar w_1-\bar z_3)\,\delta(\bar w_2-\bar z_3)}{z_{12}^{-\Delta_3}\,(w_3-z_2)^{2-\Delta_1}\,(z_1-w_3)^{2-\Delta_2}}\\ &= \frac{\delta(\beta)}{z_{12}^{-\Delta_3}\,\bar z_{31}^{1-\Delta_1}\,\bar z_{32}^{1-\Delta_2}}\int_{z_2}^{z_1}\frac{\mathrm{sgn}(z_{12})\,\mathrm{d}w_3}{(w_3-z_3)^{1-\Delta_3}\,(w_3-z_2)^{2-\Delta_1}\,(z_1-w_3)^{2-\Delta_2}}\end{aligned}$$

$$\Longrightarrow\quad \mathcal{L}_3(1^-_{\Delta_1,-1},2^-_{\Delta_2,-1},3^+_{\Delta_3,+1}) = \frac{\delta(\beta)\,B(\Delta_1-1,\Delta_2-1)\,\mathrm{sgn}(z_{12})}{z_{13}^{\Delta_1-1}\,z_{23}^{\Delta_2-1}\,\bar z_{31}^{1-\Delta_1}\,\bar z_{32}^{1-\Delta_2}}\,.$$

Parity conjugate

$$\mathcal{L}_3(1^+_{\Delta_1,+1},2^+_{\Delta_2,+1},3^-_{\Delta_3,-1}) = \frac{\delta(\beta)\,B(\Delta_1-1,\Delta_2-1)\,\mathrm{sgn}(\bar z_{12})}{z_{31}^{1-\Delta_1}\,z_{32}^{1-\Delta_2}\,\bar z_{13}^{\Delta_1-1}\,\bar z_{23}^{\Delta_2-1}}\,.$$

Observations

$$\mathcal{L}_2(1^+_{\Delta_1,+1}, 2^-_{\Delta_2,-1}) = \frac{\delta(\beta)}{z_{21}^{1-\Delta_1}\,\bar{z}_{12}^{1-\Delta_2}}\,.$$

$$\mathcal{L}_3(1^+_{\Delta_1,+1}, 2^+_{\Delta_2,+1}, 3^-_{\Delta_3,-1}) = \frac{\delta(\beta)\,B(\Delta_1-1,\Delta_2-1)\,\mathrm{sgn}(\bar{z}_{12})}{z_{31}^{1-\Delta_1}\,z_{32}^{1-\Delta_2}\,\bar{z}_{13}^{\Delta_1-1}\,\bar{z}_{23}^{\Delta_2-1}}\,.$$

- Ambidextrous prescription absorbs *all* momentum conserving delta functions.
- Brings amplitude to standard conformally covariant power law.
- Can read off light transform celestial OPE.

$$\begin{aligned}\mathbf{L}[O^a_{\Delta_1,+}](z_1,\bar{z}_1)\,\mathbf{L}[O^b_{\Delta_2,+}](z_2,\bar{z}_2)&\\ \sim \mathrm{sgn}(\bar{z}_{12})\,B(\Delta_1-1,\Delta_2-1)\,f^{abc}\,\mathbf{L}[O^c_{\Delta_1+\Delta_2-1,+}](z_2,\bar{z}_2)+\cdots&\end{aligned}$$

Graviton examples

2 points

$$\mathcal{A}_2(1^+_{\Delta_1,+2}, 2^-_{\Delta_2,-2}) = \delta(\beta)\,\delta(z_{12})\,\delta(\bar{z}_{12})$$

3 points

$$\mathcal{A}_3(1^+_{\Delta_1,+2}, 2^+_{\Delta_2,+2}, 3^-_{\Delta_3,-2}) = \delta(\beta+\mathrm{i})\,\Theta\!\left(\frac{\bar{z}_{13}}{\bar{z}_{12}}\right)\Theta\!\left(\frac{\bar{z}_{32}}{\bar{z}_{12}}\right)\frac{\delta(z_{13})\,\delta(z_{23})}{\bar{z}_{12}^{-2-\Delta_3}\,\bar{z}_{32}^{2-\Delta_1}\,\bar{z}_{13}^{2-\Delta_2}}$$

$$\boxed{\mathcal{L}_2(1^+_{\Delta_1,+2}, 2^-_{\Delta_2,-2}) = \frac{\delta(\beta)}{z_{21}^{-\Delta_1}\,\bar{z}_{12}^{-\Delta_2}}\,.}$$

$$\boxed{\mathcal{L}_3(1^+_{\Delta_1,+2}, 2^+_{\Delta_2,+2}, 3^-_{\Delta_3,-2}) = \frac{\delta(\beta+\mathrm{i})\,B(\Delta_1-1,\Delta_2-1)\,|\bar{z}_{12}|}{z_{31}^{-\Delta_1}\,z_{32}^{-\Delta_2}\,\bar{z}_{13}^{\Delta_1-1}\,\bar{z}_{23}^{\Delta_2-1}}\,.}$$

Celestial OPE

$$\mathbf{L}[G_{\Delta_1,+}](z_1,\bar{z}_1)\,\mathbf{L}[G_{\Delta_2,+}](z_2,\bar{z}_2) \sim |\bar{z}_{12}|\;B(\Delta_1-1,\Delta_2-1)\,\mathbf{L}[G_{\Delta_1+\Delta_2,+}](z_2,\bar{z}_2)+\cdots$$

Comparing celestial OPE

[Pate, Raclariu, Strominger, Yuan '19] [Fan, Fotopoulos, Taylor '19]

For boost eigenstates

$$O^a_{\Delta_1,+}(z_1,\bar{z}_1)\, O^b_{\Delta_2,+}(z_2,\bar{z}_2)$$
$$\sim \frac{1}{z_{12}}\, B(\Delta_1-1,\Delta_2-1)\, f^{abc}\, O^c_{\Delta_1+\Delta_2-1,+}(z_2,\bar{z}_2)+\cdots$$

$$G_{\Delta_1,+}(z_1,\bar{z}_1)\, G_{\Delta_2,+}(z_2,\bar{z}_2)$$
$$\sim \frac{\bar{z}_{12}}{z_{12}}\, B(\Delta_1-1,\Delta_2-1)\, G_{\Delta_1+\Delta_2,+}(z_2,\bar{z}_2)+\cdots$$

For light transformed states

$$\mathbf{L}[O^a_{\Delta_1,+}](z_1,\bar{z}_1)\, \mathbf{L}[O^b_{\Delta_2,+}](z_2,\bar{z}_2)$$
$$\sim \mathrm{sgn}(\bar{z}_{12})\, B(\Delta_1-1,\Delta_2-1)\, f^{abc}\, \mathbf{L}[O^c_{\Delta_1+\Delta_2-1,+}](z_2,\bar{z}_2)+\cdots$$

$$\mathbf{L}[G_{\Delta_1,+}](z_1,\bar{z}_1)\, \mathbf{L}[G_{\Delta_2,+}](z_2,\bar{z}_2)$$
$$\sim |\bar{z}_{12}|\, B(\Delta_1-1,\Delta_2-1)\, \mathbf{L}[G_{\Delta_1+\Delta_2,+}](z_2,\bar{z}_2)+\cdots$$

Non-singular OPE?

Toward positive geometries

(Ongoing work with Matteo Parisi and Anders Schreiber)

$N=4$ SYM super-amplitude in non-chiral superspace $(\lambda_i^\alpha, \bar{\lambda}_i^{\dot{\alpha}}, \eta_i^\alpha, \bar{\eta}_i^{\dot{\alpha}})$

$$A_{n,k} = \int_\Gamma \Omega_{n,k}(C)\; \delta^{2k|2k}(C\cdot(\bar{\lambda}\,|\,\bar{\eta}))\; \delta^{2(n-k)|2(n-k)}(C^\perp\cdot(\lambda\,|\,\eta))$$

[Arkani-Hamed, Cachazo, Cheung, Kaplan '10]

[He, Zhang '18]

- $\lambda \equiv (\lambda_i^\alpha)_{2\times n}$, etc.
- $C \in \mathrm{G}(k,n) =$ Space of k-planes in n dimensions
- $\Omega_{n,k}(C) = \dfrac{\mathrm{d}^{k\times n}C\,/\,\mathrm{vol}\,\mathrm{GL}(k)}{(1\,2\,\cdots\,k)\,(2\,3\,\cdots\,k+1)\cdots(n\,1\,\cdots\,k-1)}$
- $C^\perp \in \mathrm{G}(n-k,n)$ s.t. $C^\perp\cdot C^t = 0$
- $\Gamma \longrightarrow$ BCFW contour

Ordered minors of C labeled by columns involved

Grassmannian for celestial amplitudes

$$\mathcal{A}_{n,k} = \int_{\mathbb{R}_+^n} \frac{\mathrm{d}\omega_i}{\omega_i}\, \omega_i^{\Delta_i} \int_\Gamma \Omega_{n,k}(C)\; \delta^{2k|2k}(C\cdot(\bar\lambda\,|\,\bar\eta))\; \delta^{2(n-k)|2(n-k)}(C^\perp\cdot(\lambda\,|\,\eta))$$

Can *perform* the Mellin integrals

Can *geometrize* the Mellin integrals

- $(\lambda_i, \bar\lambda_i, \eta_i, \bar\eta_i) \equiv \sqrt{\omega_i}\,(\zeta_i, \bar\zeta_i, \chi_i, \bar\chi_i)$
- Gauge fix: $C = \left(\ I_{k\times k}\ \middle|\ (c)_{k\times n-k}\ \right)$

$$C^\perp = \left(\ (-c^t)_{n-k\times k}\ \middle|\ I_{n-k\times n-k}\ \right)$$

Performing the integrals

[AS '21] [Ferro, Moerman '21]

$$\mathcal{A}_{n,k} = \int_\Gamma \Omega_{n,k}(C) \prod_{j=1}^{n} \Theta(E_j)\, E_j^{\Delta_j - 1} \prod_{\bar{\mathsf{a}}} \delta\left(\sum_{\mathsf{b}} \epsilon_{\mathsf{b}}\, c_{\bar{\mathsf{a}}\mathsf{b}}\, \bar{z}_{\bar{\mathsf{a}}\mathsf{b}}\right) \delta^{0|2}\left(E_{\bar{\mathsf{a}}}\, \bar{\chi}_{\bar{\mathsf{a}}} + \sum_{\mathsf{b}} \epsilon_{\mathsf{b}}\, c_{\bar{\mathsf{a}}\mathsf{b}}\, \bar{\chi}_{\mathsf{b}}\right)$$
$$\times \prod_{\mathsf{a}} \delta\left(\sum_{\bar{\mathsf{b}}} \epsilon_{\bar{\mathsf{b}}}\, c_{\bar{\mathsf{b}}\mathsf{a}}\, z_{\mathsf{a}\bar{\mathsf{b}}}\right) \delta^{0|2}\left(E_{\mathsf{a}}\, \chi_{\mathsf{a}} - \sum_{\bar{\mathsf{b}}} \epsilon_{\bar{\mathsf{b}}}\, c_{\bar{\mathsf{b}}\mathsf{a}}\, \chi_{\bar{\mathsf{b}}}\right)$$

$$C_{\bar{\mathsf{a}}\bar{\mathsf{b}}} = \delta_{\bar{\mathsf{a}}\bar{\mathsf{b}}}\,, \qquad C_{\bar{\mathsf{a}}\mathsf{a}} = c_{\bar{\mathsf{a}}\mathsf{a}}$$

Gauge choice

$i = \mathsf{a}, \bar{\mathsf{a}}$

$\bar{\mathsf{a}} = 1, 2, \ldots, k$

"Negative helicity"

$\mathsf{a} = k+1, k+2, \ldots, n$

"Positive helicity"

$$E_{\bar{\mathsf{a}}} = \omega_{\bar{\mathsf{a}}} = -\sum_{\mathsf{b}} \epsilon_{\mathsf{b}}\, c_{\bar{\mathsf{a}}\mathsf{b}}$$
$$E_{\mathsf{a}} = \omega_{\mathsf{a}} = \sum_{\bar{\mathsf{b}}} \epsilon_{\bar{\mathsf{b}}}\, c_{\bar{\mathsf{b}}\mathsf{a}}$$

$$(\lambda_i, \bar{\lambda}_i, \eta_i, \bar{\eta}_i) \equiv \sqrt{\omega_i}\, (\zeta_i, \bar{\zeta}_i, \chi_i, \bar{\chi}_i)$$

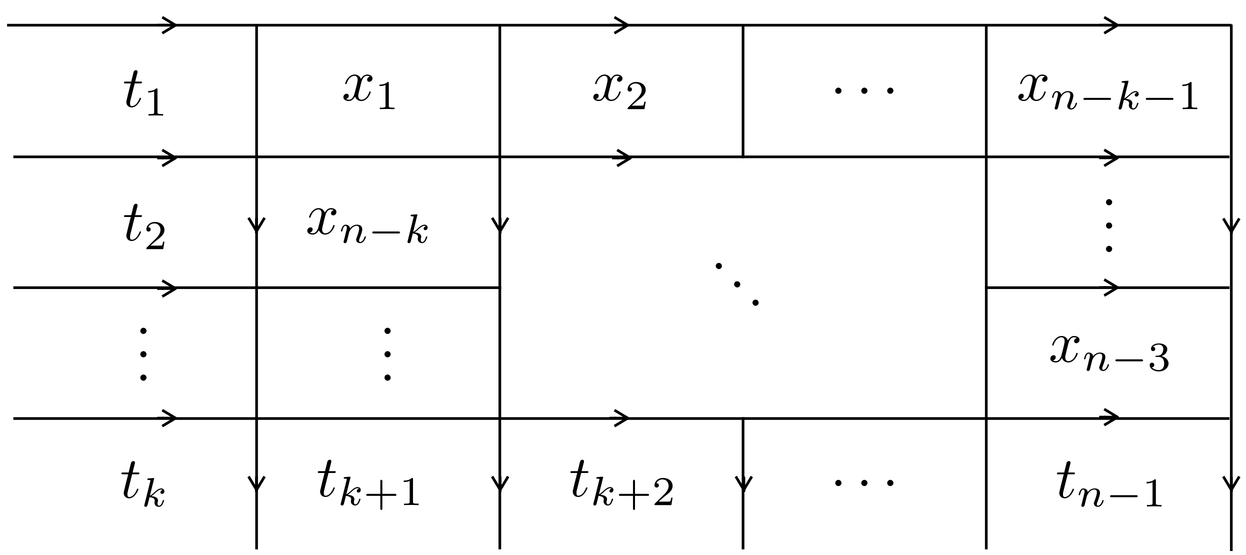
Geometrizing the integrals

Upcoming: [Parisi, Schreiber, AS]

$$\mathcal{A}_{n,k} = \delta\left(\mathrm{i}\sum_{i=1}^{n}\Delta_i\right)\int_{\Gamma}\prod_{i=1}^{n-1}\Theta\left(\frac{\bar{t}_i}{t_i}\right)\frac{\mathrm{d}t_i}{t_i^{1+\sum_{p=1}^{i}\Delta_p}}\,\frac{\mathrm{d}\bar{t}_i}{\bar{t}_i^{1+\sum_{p=i+1}^{n}\Delta_p}}\prod_{e=1}^{n-3}\frac{\mathrm{d}x_e}{x_e}$$
$$\times\,\delta^{2k|2k}\big(C(t)\cdot(\bar{\zeta}\,|\,\bar{\chi})\big)\,\delta^{2(n-k)|2(n-k)}\big(C(\bar{t})^{\perp}\cdot(\zeta\,|\,\chi)\big)$$

Web parametrisation of C with two copies of torus actions

[Speyer, Williams '05]



$$C_{\bar{\mathsf{a}}\bar{\mathsf{b}}}(t) = \delta_{\bar{\mathsf{a}}\bar{\mathsf{b}}}$$
$$C_{\bar{\mathsf{a}}\mathsf{a}}(t) = (-1)^{\bar{\mathsf{a}}+k}\prod_{i=\bar{\mathsf{a}}}^{\mathsf{a}-1} t_i \sum_{p:\bar{\mathsf{a}}\to\mathsf{a}} \mathrm{Prod}_p(x)$$

$\bar{\mathsf{a}} = 1, 2, \ldots, k \qquad \mathsf{a} = k+1, k+2, \ldots, n$

Future directions

- Find the right basis of conformal primary local operators.

- Determine light and shadow transform OPEs in generality.

- Clarify twistorial origins of $w_{1+\infty}$ symmetries in a light transform basis.

  [Strominger '21]

- Build bridges with positive geometries.