Aspects of Celestial MHV gluon Amplitudes

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A new basis for the S-matrix.

$$p_i = \epsilon_i \omega_i \left(1 + z_i \bar{z}_i, z_i + \bar{z}_i, -i(z_i - \bar{z}_i), 1 + z_i \bar{z}_i\right)$$

$z_i, \bar{z}_i$ are complex coordinates on the celestial sphere.
The Celestial amplitude

$$\tilde{\mathcal{A}}_n \left(\Delta_i, z_i, \bar{z}_i\right) = \int \frac{d\omega_1}{\omega_1} \dots \frac{d\omega_n}{\omega_n} \omega_1^{\Delta_1} \dots \omega_n^{\Delta_n} \mathcal{A}_n(\omega_i, z_i, \bar{z}_i)$$

transforms as a correlation function of n operators with (Δ_i, J_i).

$$\tilde{\mathcal{A}}_n \left(\Delta_i, \frac{a z_i + b}{c z_i + d}, \frac{\bar{a}\bar{z}_i + \bar{b}}{\bar{c}\bar{z}_i + \bar{d}}\right) = \prod_{i=1}^{n} \left[(c z_i + d)^{\Delta_i + J_i} \left(\bar{c}\bar{z}_i + \bar{d}\right)^{\Delta_i - J_i}\right] \tilde{\mathcal{A}}_n \left(\Delta_i, z_i, \bar{z}_i\right)$$

Pasterski, Shao, Strominger

(Conformally) soft limits

$$\lim_{\Delta_i \to -k} (\Delta_i + k)\tilde{\mathcal{A}}_n = \int d\omega_i \left(\lim_{\Delta_i \to -k} (\Delta_i + k)\omega_i^{\Delta_i + k - 1} \right) \omega_i^{-k} \mathcal{A}_n$$
$$= Res_{\omega_i \to 0} \omega_i^{-k-1} \mathcal{A}_n$$

Donnay, Puhm, Strominger

Celestial amplitudes have an infinite number of poles as $\Delta_i \to -k$ for $k \in \mathbb{Z}$

The gluon operators turn into currents in this limit

$$\lim_{\Delta_i \to -k} (\Delta_i + k)\mathcal{O}^{+,a} = R^{k,a}$$

The infinite number of soft currents form a closed algebra.

Guevara, Himwich, Pate, Strominger

Celestial amplitudes exhibit interesting behaviour in the $\Delta \to -k$ limit.

Part 1: Soft singularities and polytopes

Based on work in progress with Lecheng Ren, Mark Spradlin and Anastasia Volovich

Part 2: Soft singularities and differential equations

Based on 2106.16111 with Yangrui Hu, Lecheng Ren and Anastasia Volovich

Soft singularities and polytopes

Positive geometries

Scattering amplitudes in momentum space have been reformulated in terms of positive geometries for some theories

The Amplituhedron in $\mathcal{N} = 4$ SYM and the Associahedron in biadjoint Tr[ϕ^3] theory

Arkani-Hamed, Bai, He, Yan, Trnka

Positive geometries also appear in EFTs, CFTs, cosmology

Arkani-Hamed, Baumann, Benincasa, Huang, Lee, Pimentel, Postnikov, Shao

Tree-level Feynman diagrams in Tr[ϕ^3] are in a one to one correspondence with the triangulations of a polygon.

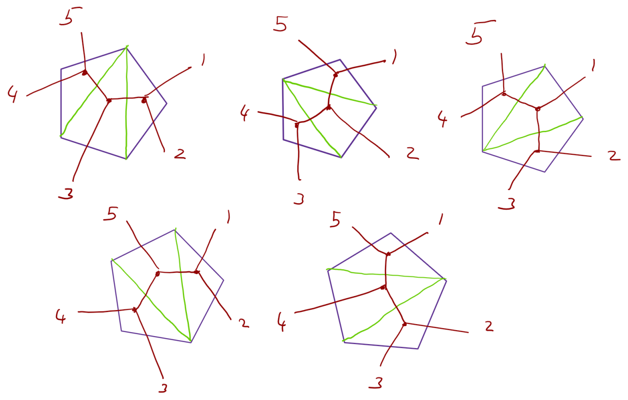


Figure 1: $n = 5$ with orderings $(1234|1234)$

Arkani-Hamed, Bai, He, Yan

Associahedron

The (complete) triangulations of a polygon are the vertices of a polytope called the Associahedron or Stasheff polytope

Stasheff, Tamari

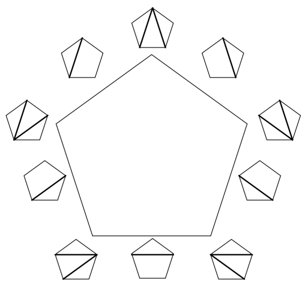
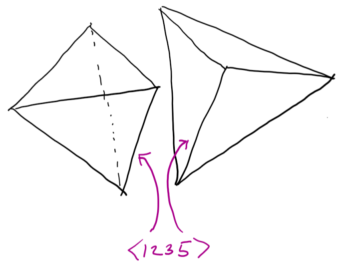


Figure 2: Associahedron for $n = 5$

This dictates what poles can appear together in the amplitude

$\mathcal{N} = 4$ super Yang-Mills

$$\mathcal{A}_6\left(1^-, 2^-, 3^-, 4^+, 5^+, 6^+\right)$$



$$= \frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 1235\rangle} + \frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 1236\rangle}$$

Hodges

The amplitude can be interpreted as the volume of the tetrahedron
The poles in the amplitude correspond to the faces of the tetrahedron

Singularities and geometry

The singularities of amplitudes are intimately tied to the facet structure of geometries

In momentum space amplitudes, can be interpreted as *dlog* forms on the geometries

Interesting to understand what role geometries play in Celestial amplitudes.

Celestial amplitudes have two distinct types of singularities singularities in Δ and singularities in z

The focus of this section will be on the singularities in Δ in MHV tree level gluon amplitudes

MHV amplitudes in momentum space

The tree level MHV amplitude is given by the Parke-Taylor formula

$$\mathcal{A}_n\left(1^-, 2^-, 3^+, \dots n^+\right) \;=\; \frac{\langle 12\rangle^4\, \delta^{(4)}\left(\sum_{i=1}^n p_i\right)}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle} \quad .$$

A parametrization suitable for celestial amplitudes is

$$\lambda_i^{\alpha} \;= \epsilon_i\,\sqrt{2\omega_i}\begin{pmatrix}1\\ z_i\end{pmatrix} \;,\; \lambda_{i,\alpha} \;=\; \epsilon_i\,\sqrt{2\omega_i}\begin{pmatrix}-z_i\\ 1\end{pmatrix} \quad ,$$

$$\tilde{\lambda}_{i,\dot{\alpha}} \;= \sqrt{2\omega_i}\begin{pmatrix}-\bar{z}_i\\ 1\end{pmatrix} \;,\; \tilde{\lambda}_i^{\dot{\alpha}} \;=\; \sqrt{2\omega_i}\begin{pmatrix}1\\ \bar{z}_i\end{pmatrix} \quad ,$$

$$\langle i\, j\rangle \;=\; -\,2\epsilon_i\epsilon_j\sqrt{\omega_i\omega_j}\, z_{ij} \;,\; [i\, j] \;=\; 2\sqrt{\omega_i\omega_j}\,\bar{z}_{ij} \quad .$$

Mellin transform of the MHV amplitudes

$$\tilde{\mathcal{A}}_n\left(1^-, 2^-, \ldots, n^+\right) = \frac{z_{12}^4}{z_{12}z_{23}\ldots z_{n1}} \int \prod_{i=1}^{n} \frac{d\omega_i}{\omega_i} \omega^{\Delta_i} \left(\frac{\omega_1\omega_2}{\omega_3\ldots\omega_n}\right) \delta^{(4)}\left(\sum_{i=1}^{n} p_i\right)$$

Integrate out an overall scale produces a delta function

$$\int \frac{d\omega_1}{\omega_1} \omega_1^{\left(\sum_{i=1}^{n}\Delta_i - 1\right)} = \delta\left(\sum_{i=1}^{n}\left(\Delta_i - 1\right)\right)$$

$$\mathcal{A}_n\left(t\,1^-, t\,2^-, \ldots, t\,n^+\right) = t^{-n}\mathcal{A}_n\left(1^-, 2^-, \ldots, n^+\right)$$

Mellin transform of the MHV amplitudes

We use the momentum conserving delta function to solve for $\omega_{n-3}, \omega_{n-2}, \omega_{n-1}, \omega_n$.

$$\delta^{(4)}\left(\sum_{i=1}^{n} p_i\right) = \frac{1}{U}\prod_{b=n-3}^{n} \delta\left(\omega_b - \omega_b^*\right)\Theta\left(\omega_b^*\right)$$

$$\omega_b^* = \sum_{a=1}^{n-3} x_{a,b}\,\omega_a$$

where

$$p_i = \epsilon_i \omega_i q_i(z_i, \bar{z}_i) \qquad x_{a,b} = -\frac{\epsilon_a\, U_{a,b}}{\epsilon_b\, U}$$

$$U = \det\{q_{n-3}, q_{n-2}, q_{n-1}, q_n\},\ U_{a,b} = \det\{q_{n-3}, q_{n-2}, q_{n-1}, q_n\}|_{b\to a}$$

n>4 point MHV amplitudes

$$\tilde{\mathcal{A}}_n^{\text{MHV}} = \mathcal{N}(z_i, \bar{z}_i) \prod_{a,b} \Theta(x_{a,b})\ \delta\left(\sum_i (\Delta_i - 1)\right) \times$$
$$\int \left(\prod_{c=2}^{n-4} \frac{du_c}{u_c}\right) \left(\prod_{a=2}^{n-4} u_a^{\Delta_a - J_a}\right) \prod_{b=n-3}^{n} \left(\sum_{a=2}^{n-4} x_{a,b} u_a + x_{1,b}\right)^{\Delta_b - J_b - 1}$$

Schreiber, Volovich, Zlotnikov

This is a multidimensional Mellin transform whose domain of convergence is governed by a specific polytope

Convergence of Mellin integrals

$$I(\Delta_i, z_i) = \int \prod_{i=1}^{n} \frac{d\omega_i}{\omega_i} \omega_i^{\Delta_i} \frac{1}{f(\omega_i)}$$

The integral converges whenever $\Delta = (\Delta_1, \ldots \Delta_n)$ lies in the interior of the Newton polytope of f.

$$0 \in \mathrm{N}(f) - \Delta$$

It can be shown that I has an infinite number of poles (in Δ) emanating from the facets of the Newton polytope.

It can be analytically continued past these poles.

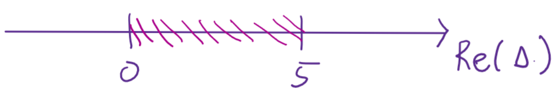
Nilsson, Passare

Newton Polytope -1D

$$f(z) = \sum_{\Delta \in A} a_\Delta z^\Delta \qquad a_\Delta \in \mathbb{C}^*, \qquad z \in (\mathbb{C}^*)^n$$

The Newton polytope of $f(z)$, denoted by $\mathbf{N}(f)$ is the Convex hull of A in $\mathbb{R}^n$

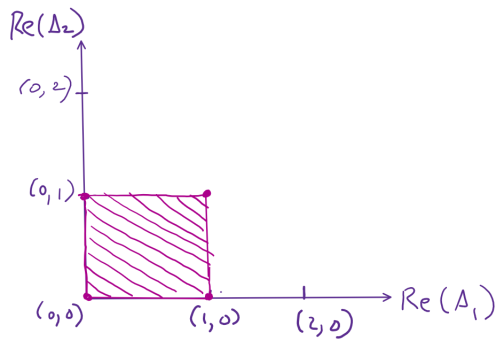
$$f(z) = a_0 + a_2 z^2 + a_5 z^5$$



The Newton polytope of f is the interval $(0, 5)$.

Newton Polytope - 2D

$$f(z_1, z_2) = a_{0,0} + a_{1,0}z_1 + a_{0,1}z_2 + a_{2,2}z_1z_2$$



The Newton polytope of f is the square with vertices $(0,0), (0,1), (1,0), (1,1)$.

Convergence of Mellin integrals - example 1

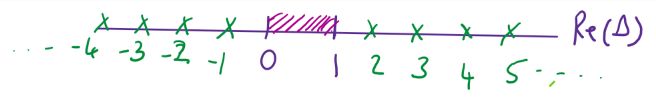
Let $f(z) = a + bz$. Its Mellin transform

$$\tilde{f}(\Delta) = \int_0^\infty \frac{dz}{z} z^\Delta \frac{1}{a + b\,z}$$

Converges for $Re(\Delta) \in (0, 1)$. It can be analytically continued to

$$\tilde{f}(\Delta) = a^{\Delta - 1} b^{-\Delta} \Gamma(\Delta)\Gamma(1 - \Delta)$$

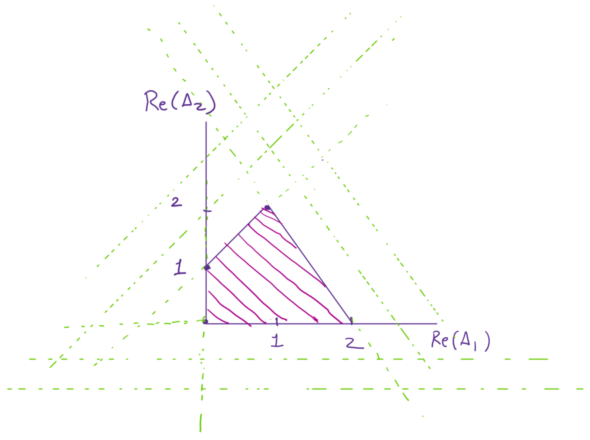
Poles emanating from endpoints of $\mathsf{N}(f) = (0, 1)$



Convergence of Mellin integrals - example 2

Let $f(z) = 1 + z_2 + z_1^2 + z_1 z_2^2$. Its Mellin transform

$$\tilde{f}(\Delta) = \int_0^\infty \frac{dz_1}{z_1}\frac{dz_2}{z_2}\, z_1^{\Delta_1} z_2^{\Delta_2} \frac{1}{1 + z_2 + z_1^2 + z_1 z_2^2}$$



Convergence of Mellin integrals - example 2

The equations of the facets are

$$Re(\Delta_1) \geq 0 \qquad Re(\Delta_1) - Re(\Delta_2) \geq -1$$
$$-2Re(\Delta_1) - Re(\Delta_2) \geq -4 \qquad Re(\Delta_2) \geq 0$$

The analytic continuation will be of the form

$$\Gamma(\Delta_1)\,\Gamma(\Delta_1 - \Delta_2 + 1)\,\Gamma(4 - 2\Delta_1 - \Delta_2)\,\Gamma(\Delta_2)\,\Phi$$

In general, the arguments of the Gamma functions are determined by the facets of the Newton Polytope. The integral has an analytic continuation

$$I(\Delta_i, z_i) = \Phi(\Delta_i, z_i) \prod_{k=1}^{N_f} \Gamma(n_k \cdot \Delta - m_k)\,.$$

The analytic continuation has poles on all hyperplanes emanating from the Newton polytope. Here Φ is an entire function.

$$\tilde{\mathcal{M}}_6^{\text{tree}} \sim \delta\left(\sum_{i=1}^{6}(\Delta_i - 1)\right)\int \frac{du_2}{u_2}\; u_2^{\Delta_2 - J_2}\prod_{b=3}^{6}\left(x_{2,b}u_2 + x_{1,b}\right)^{\Delta_b - J_b - 1}$$

The integral converges when

$$Re(\Delta_2 - J_2) \in \left(0,\; \sum_{b=3}^{6} Re(-\Delta_b + J_b + 1))\right)$$

The Gamma function that appear are

$$\begin{aligned}&\Gamma\left(\Delta_2 - J_2\right)\Gamma\left(\sum_{b=3}^{6} Re(-\Delta_b + J_b + 1) - (\Delta_2 - J_2 - 1)\right)\\ =&\,\Gamma\left(\Delta_2 - J_2\right)\Gamma\left(\Delta_1 - J_1\right)\end{aligned}$$

Generic MHV tree

The analysis can be extended to any number of points for MHV amplitudes.

The polytope defining the region of convergence is a simplex with sides of length

$$\sum_{b=n-3}^{n} Re(1 + J_b - \Delta_b)$$

This implies that the MHV amplitude has singularities given by

$$\prod_{i=1}^{n-4} \Gamma(\Delta_i - J_i)$$

Summary of part 1

We analyzed the domain of convergence of Mellin integrals.

The interior of the appropriate Newton polytope is the domain of convergence of the Mellin integrals. It also dictates the pole structure of their analytic continuations.

We applied this analysis to celestial MHV gluon amplitudes and discovered that the n-point MHV amplitude is a simplex.

The singularities corresponding to the facets of this simplex are the conformally soft singularities of the amplitude.

Soft singularities and differential equations

Hypergeometric differential equations

The behaviour of celestial amplitudes in the conformally soft limit implies the existence of nontrivial differential equations.

$$\tilde{\mathcal{A}}_n^{\text{MHV}} = \mathcal{N}\left(z_i, \bar{z}_i\right) \prod_{a,b} \Theta\left(x_{a,b}\right) \, \delta\left(\sum_i (\Delta_i - 1)\right) \times$$
$$\int \left(\prod_{c=2}^{n-4} \frac{du_c}{u_c}\right) \left(\prod_{a=2}^{n-4} u_a^{\Delta_a - J_a}\right) \prod_{b=n-3}^{n} \left(\sum_{a=2}^{n-4} x_{a,b} u_a + x_{1,b}\right)^{\Delta_b - J_b - 1}$$

These are known to be integral representations of Aomoto-Gelfand hypergeometric functions.
They satisfy the multivariate generalizations of the hypergeometric differential equations.

Hypergeometric differential equations

There are n hypergeometric differential equations. Four of these are a direct consequence of momentum conservation

$$\sum_{i=1}^{n} \epsilon_i \, q_i^{\mu} e^{\frac{\partial}{\partial \Delta_i}} \tilde{\mathcal{A}}_n^{\text{MHV}} = 0 \quad .$$

The momentum operator acts non trivially because we have integrated out the delta function.

$$\sum_{a=1}^{n-4} x_{a,b} \frac{\partial \tilde{\mathcal{A}}_n^{\text{MHV}}}{\partial x_{a,b}} = (\Delta_b - J_b - 1) \, \tilde{\mathcal{A}}_n^{\text{MHV}}$$

$$b = 1, \ldots 4$$

The remaining $n-4$ arise as a consequence of $GL(n-4)$ transformations on $x_{a,b}$.

$$\tilde{\mathcal{A}}_n^{\text{MHV}} = \mathcal{N}(z_i, \bar{z}_i) \prod_{a,b} \Theta\left(x_{a,b}\right) \delta\left(\sum_i (\Delta_i - 1)\right) \times$$
$$\int \left(\prod_{c=2}^{n-4} \frac{du_c}{u_c}\right) \left(\prod_{a=2}^{n-4} u_a^{\Delta_a - J_a}\right) \prod_{b=n-3}^{n} \left(\sum_{a=2}^{n-4} x_{a,b} u_a + x_{1,b}\right)^{\Delta_b - J_b - 1}$$

This is the result of using the momentum conserving delta function to eliminate $\omega_{n-3}, \ldots \omega_n$.

A different choice would lead to a similar integral with different $x_{a,b}$

These are equivalent upto a $GL(n-4)$ transformation.

Hypergeometric differential equations

There are $n-4$ differential equations that encode the behaviour of the amplitude under these $GL(n-4)$ transformations.

$$\sum_{b=n-3}^{n} x_{a,b} \frac{\partial \tilde{\mathcal{A}}_n^{\text{MHV}}}{\partial x_{a,b}} = -\left(\Delta_a - J_a\right) \tilde{\mathcal{A}}_n^{\text{MHV}}$$

This completes the set of hypergeometric differential equations.

There are other differential equations that arise from the conformally soft behaviour of celestial amplitudes whose momentum space origins are more obscure.

Banerjee, Ghosh

Leading soft current

MHV amplitudes are closed under soft limits

$$j^{+,a}(z) = \lim_{\Delta \to 1} (\Delta - 1)\, \mathcal{O}^{+,a}(z,\bar{z})$$

The leading soft current $j^{+,a}$ is a Kac-Moody current and satisfies a Ward identity

$$\left\langle j^{+,a}(z) \prod_{i=1}^{n} \mathcal{O}^{+,a_i}(z_i,\bar{z}_i) \right\rangle = -\sum_{k=1}^{n} \frac{T_k^a}{z - z_k} \left\langle \prod_{i=1}^{n} \mathcal{O}^{+,a_i}(z_i,\bar{z}_i) \right\rangle$$

This is a restatement of the soft gluon theorem.

He, Mitra, Strominger

$$S^{+,a}(z,\bar{z}) = \lim_{\Delta\to 0} \Delta \mathcal{O}^{+,a}(z,\bar{z})$$

has its own Ward identity

$$\begin{aligned} &\left\langle S^{+,a}(z,\bar{z}) \prod_{i=1}^{n} \mathcal{O}^{+,a_i}(z_i,\bar{z}_i) \right\rangle \\ &= -\sum_{k=1}^{n} \frac{\epsilon_k}{z-z_k} \left(-2\bar{h}_k + 1 + (\bar{z}-\bar{z}_k)\bar{\partial}_k\right) T_k^a P_k^{-1} \left\langle \prod_{i=1}^{n} \mathcal{O}^{+,a_i}(z_i,\bar{z}_i) \right\rangle \end{aligned}$$

This is also the subleading soft gluon theorem.

Constraint on OPE

We can extract the OPE between $S^{+,a}(z,\bar{z})$ and a hard gluon $\mathcal{O}^{+,a}(z_1,\bar{z}_1)$from the ward identity

$$S^{+,a}(z,\bar{z})\,\mathcal{O}^{+,a}(z_1,\bar{z}_1) \sim \left[\frac{1}{z-z_1}(\dots) + \sum_{p=1}^{\infty}(z-z_1)^{p-1}(\dots)\right]\mathcal{O}^{+,a}(z_1,\bar{z}_1)$$
$$+ (\bar{z}-\bar{z}_1)\left[\frac{1}{z-z_1}(\dots) + \sum_{p=1}^{\infty}(z-z_1)^{p-1}(\dots)\right]\mathcal{O}^{+,a}(z_1,\bar{z}_1)$$

However, the OPE of two hard gluons is determined by asymptotic symmetries/collinear limits

Constraints on OPE

$$\mathcal{O}^{+,a}_{\Delta_1}(z,\bar{z})\mathcal{O}^{+,b}_{\Delta_2}(z_1,\bar{z}_1) \sim -iB(\Delta-1,\Delta_1-1)\left[\frac{f^{abc}}{z-z_1}+\frac{\Delta-1}{\Delta+\Delta_1-2}f^{abc}L_{-1}\right.$$
$$\left.+i\frac{\Delta-1}{\Delta+\Delta_1-2}\left(\delta^{ac}\delta^{be}+\delta^{ae}\delta^{bc}\right)j^e_{-1}\right]\mathcal{O}^{+,c}_{\Delta+\Delta_1-1}(z_1,\bar{z}_1)$$

Ebert, Fan, Fotopoulos, Pate, Raclariu, Sharma, Strominger, Taylor, Wang, Yuan

The OPE of the subleading soft current with a hard gluon can be computed by

$$S^{+,a}(z,\bar{z})\,\mathcal{O}^{+,a}(z_1,\bar{z}_1)=\lim_{\Delta\to 0}\Delta\,\mathcal{O}^{+,a}_{\Delta}(z,\bar{z})\mathcal{O}^{+,b}_{\Delta_1}(z_1,\bar{z}_1)$$

Null state equation

Equating the two gives us

$$\Psi \equiv \mathcal{D}\mathcal{O}^{+,a}_{\Delta_1}(z_1, \bar{z}_1) = 0$$

Ψ is a null state

$$L_1\Psi = \bar{L}_1\Psi = j^{+,a}_m\Psi = 0$$

Inserting this into a correlation function yields the differential equation

$$\langle \Psi \mathcal{O}^{a_2}_{\Delta_2} \ldots \mathcal{O}^{a_i}_{\Delta_i} \ldots \mathcal{O}^{a_n}_{\Delta_n} \rangle = 0$$

Null State equation

$$
\begin{aligned}
&T_i^a \frac{\partial}{\partial z_i} \langle \mathcal{O}^{a_1}_{\Delta_1} \dots \mathcal{O}^{a_i}_{\Delta_i} \dots \mathcal{O}^{a_n}_{\Delta_n} \rangle \\
&- \sum_{j \neq i} \epsilon_i \epsilon_j \frac{\Delta_j - J_j - 1 + \bar{z}_{ji} \bar{\partial}_j}{z_{ji}} T_j^a \langle \mathcal{O}^{a_1}_{\Delta_1} \dots \mathcal{O}^{a_j}_{\Delta_{j-1}} \dots \mathcal{O}^{a_i}_{\Delta_{i+1}} \dots \mathcal{O}^{a_n}_{\Delta_n} \rangle \\
&- \sum_{j \neq i} \frac{T_j^{a_i}}{z_{ji}} \langle \mathcal{O}^{a_1}_{\Delta_1} \dots \mathcal{O}^{a}_{\Delta_i} \dots \mathcal{O}^{a_n}_{\Delta_n} \rangle \; + \; \Delta_i \sum_{j \neq i} \frac{T_j^a}{z_{ji}} \langle \mathcal{O}^{a_1}_{\Delta_1} \dots \mathcal{O}^{a_i}_{\Delta_i} \dots \mathcal{O}^{a_n}_{\Delta_n} \rangle = 0 \quad .
\end{aligned}
$$

Banerjee, Ghosh

One equation for each positive helicity gluon

No obvious momentum space origin.

Easier to deal with colour ordered amplitudes

Colour ordered amplitudes

Colour decomposition

$$A_n = \sum_{\pi \in S_{n-1}} \mathcal{A}_n[1, \pi(2), \pi(3), \ldots, \pi(n)] \,\mathrm{Tr}[T^{a_1} T^{a_{\pi(2)}} \cdots T^{a_{\pi(n)}}] \;\;,$$

Gluon amplitudes can be decomposed into a basis of colour ordered amplitudes

Each colour ordered amplitude is gauge invariant and is given by the Parke-Taylor formula

Null state equations for colour ordered amplitude

The null state equation reduces to a simpler form on colour ordered amplitudes.

For the canonical ordering $1, \dots n$, we get

$$\left(\partial_i - \frac{\Delta_i}{z_{i-1,i}} - \frac{1}{z_{i+1,i}}\right)\tilde{\mathcal{A}}_n(1, \cdots, n) + \left(\epsilon_i \epsilon_{i-1} \frac{\Delta_{i-1} - J_{i-1} - 1 + \bar{z}_{i-1,i}\bar{\partial}_{i-1}}{z_{i-1,i}} e^{\frac{\partial}{\partial \Delta_i} - \frac{\partial}{\partial \Delta_{i-1}}}\right)\tilde{\mathcal{A}}_n(1, \cdots, n) = 0 \ ,$$

Null state equation from BCFW shifts

$$\left(\partial_i \; - \; \frac{\Delta_i}{z_{i-1,i}} \; - \; \frac{1}{z_{i+1,i}}\right) \tilde{\mathcal{A}}_n(1, \cdots, n)$$
$$+ \; \left(\epsilon_i \epsilon_{i-1} \, \frac{\Delta_{i-1} - J_{i-1} - 1 + \bar{z}_{i-1,i} \bar{\partial}_{i-1}}{z_{i-1,i}} e^{\frac{\partial}{\partial \Delta_i} - \frac{\partial}{\partial \Delta_{i-1}}}\right) \tilde{\mathcal{A}}_n(1, \cdots, n) \; = \; 0 \quad ,$$

We can rewrite this in momentum space compactly as

$$\left(\lambda^{\alpha}_{i-1} \frac{\partial}{\partial \lambda^{\alpha}_i} - \tilde{\lambda}^{\dot{\alpha}}_i \frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}_{i-1}}\right) \mathcal{A}_n = \frac{\langle i-1\, i+1 \rangle}{\langle i+1\, i \rangle} \mathcal{A}_n \quad ,$$

The LHS implements an infinitesimal of λ_i and$\tilde{\lambda}_{i-1}$.

$$\lambda_i \to \hat{\lambda}_i = \lambda_i + \epsilon \lambda_{i-1} \qquad \tilde{\lambda}_{i-1} \to \hat{\tilde{\lambda}}_{i-1} = \tilde{\lambda}_{i-1} - \epsilon \tilde{\lambda}_i$$

BCFW shift

$$\lambda_i \to \hat{\lambda}_i = \lambda_i + \epsilon \lambda_{i-1} \qquad \tilde{\lambda}_{i-1} \to \hat{\tilde{\lambda}}_{i-1} = \tilde{\lambda}_{i-1} - \epsilon \tilde{\lambda}_i$$

We recognize this as an infinitesimal BCFW shift.

It preserves momentum conservation

$$\lambda_i \hat{\tilde{\lambda}}_{i-1} + \hat{\lambda}_i \tilde{\lambda}_i = \lambda_{i-1} \tilde{\lambda}_{i-1} + \lambda_i \tilde{\lambda}_i$$

Summary of part 2

We identified the momentum space origins for hypergeometric equations satisfied by celestial amplitudes.

We derived the null state equations satisfied by colour ordered amplitudes

These equations were seen to correspond to infinitesimal BCFW shifts.

Outlook

- Extensions of the polytopal structure of singularities in Δ extend beyond the MHV tree sector.
- Can we “import” any of the geometric structures from momentum space amplitudes?
- Is there a more fundamental principle which explains the connection between BCFW shifts and null state equations?
- What is the structure of the conformally soft limit beyond the MHV sector? Understand the interplay between positive and negative helicities.

Thank you for your attention!